Problem 2-1 (Recursive Inseparability) 20 points

Recall that two disjoint languages $A$ and $B$ are called \textit{recursively inseparable}, if there is no recursive language $R$ such that $A \subseteq R$ and $B \cap R = \emptyset$.

(a) Show that if disjoint $A, B \in \text{co} \mathcal{RE}$, then they are recursively separable.

(b)∗ Given a language $L \in \mathcal{RE}$, let $TM_L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = L\}$. Show that if $L_1$ and $L_2$ are distinct recursively enumerable languages and $L_1 \subset L_2$, then $TM_{L_1}$ is recursively inseparable from $TM_{L_2}$.

\footnote{Do you think the condition $L_1 \subset L_2$ is important? Try to see where you use it.}

\textbf{(Hint:} Notice that this implies that $TM_\emptyset = E_{TM}$ is recursively inseparable from $TM_L$, for any $L \neq \emptyset$. Does it ring a bell? Recall Rice’s theorem and try to do something analogous.)

Problem 2-2 (Completeness and Reductions) 10 points

Assume $A \in \mathcal{RE}$ is $m$-complete. For each of the following 3 conditions, say if the appropriate language $B$ exists, and if yes, what is the most general thing you say about such $B$ (i.e., which class it must/must not belong to):

(a) $A \leq_m B$ and $A \leq_m \overline{B}$;

(b) $B \leq_m A$ and $\overline{B} \leq_m A$;

(c) $B \leq_m A \leq_m \overline{B}$.

Problem 2-3 (True or False) 20 points

Let $W$ be a vocabulary. If $\Gamma$ is a set of sentences, let $\text{Thm}(\Gamma) = \{\gamma \mid \Gamma \models \gamma\}$. If $M$ is a model, let $\text{True}(M) = \{\gamma \mid \models_M \gamma\}$. Briefly answer the following questions. Be sure to give a convincing reason for your answer.

(a) Assume $\Gamma$ is consistent. Can $\text{Thm}(\Gamma)$ be recursive?

(b) Assume $\Gamma$ is finite. Does it imply that $\text{Thm}(\Gamma)$ is recursive?

(c)∗ Assume $\Gamma$ is r.e. Is there always a recursive set $\Delta$ such that $\text{Thm}(\Gamma) = \text{Thm}(\Delta)$?

\textbf{(Hint:} Recall the notion of “lexicographic enumerators” from homework 1.)
(d) We know in general that \( \text{Thm}(\Gamma) \) is r.e. when \( \Gamma \) is r.e. Assume that for every \( \gamma \), either \( \Gamma \models \gamma \) or \( \Gamma \models \neg \gamma \) (or possibly both). Does it follow that \( \text{Thm}(\Gamma) \) is recursive? 

\textbf{(Hint: Do not forget to consider two cases.)}

(e) Assume \( \text{True}(M) \) is recursive. Does it follows that some \( \Gamma \) axiomatizes \( M \), i.e. \( \text{True}(M) = \text{Thm}(\Gamma) \)?

(f)** \textit{Extra Credit:} Assume \( \Gamma \) is consistent and \( \text{Thm}(\Gamma) \) is recursive. Does it follows that \( \Gamma \) axiomatizes some model \( M \), i.e. \( \text{True}(M) = \text{Thm}(\Gamma) \)?

\textbf{Problem 2-4 (Gödel Incompleteness Again) \hspace{1cm} 25 points}

Let \( W \) be some vocabulary containing constant symbol 0 and unary function symbol \( s \) (meant to represent successor). Then we can “encode” an integer \( n \geq 0 \) in \( W \) by \( n = s(\ldots s(0)\ldots) \), where \( s \) is applied \( n \) times. Let \( \Gamma \) be a consistent (i.e., one cannot derive contradictions from \( \Gamma \)) recursively enumerable set of axioms over \( W \). Now given a formula \( \alpha(x) \) over \( W \), we say that is \textit{represents} a set of integers \( R(\alpha; \Gamma) = \{ n \mid \alpha(n) \in \text{Thm}(\Gamma) \} \). In other words, \( R(\alpha; \Gamma) \) consists of integers “making” \( \alpha \) a consequence of \( \Gamma \).

(a) Show that \( R(\alpha; \Gamma) \) is r.e.

(b) Assume \( R(\alpha; \Gamma) \) is not recursive. Show there exists an integer \( n_0 \) such that \( \Gamma \not\models \alpha(n_0) \) and \( \Gamma \not\models \neg \alpha(n_0) \). In other words, neither \( \alpha(n_0) \), nor \( \neg \alpha(n_0) \) is a valid consequence of \( \Gamma \).

\textbf{(Hint: Assuming that such \( n_0 \) does not exist, show that \( R(\alpha; \Gamma) \) is recursive, which is a contradiction.)}

(c)** Assume further that \( R(\alpha; \Gamma) \) is shown undecidable via a \textit{mapping} reduction from \( A_{\text{TM}} \): \( A_{\text{TM}} \leq_m R(\alpha; \Gamma) \). Using recursion theorem, give an explicit integer \( n_0 \) satisfying the claim of part (b).

\textbf{(Hint: Do exactly what we did to give explicit sentence which was true but not provable for integers. Of course, justify each step.)}

\textbf{Problem 2-5 (Arithmetic Hierarchy) \hspace{1cm} 15 points}

Place the following languages as low in the arithmetic hierarchy as you can (\textit{extra credit:} can you prove that you cannot go any lower?)

(a) \( \text{INF} = \{ \langle M \rangle \mid L(M) \text{ is infinite} \} \).

(b) \( \text{REG} = \{ \langle M \rangle \mid L(M) \text{ is regular} \} \).

(c) Give an explicit Turing reduction showing that \( \text{INF} \leq_t \text{REG} \).
Problem 2-6* (Post’s Theorem for \( n = 2 \))

I stated without formal proof Post’s theorem: \( \Sigma_n = \exists^n \), \( \Pi_n = \forall^n \). In this problem, you will prove the “interesting” part of this theorem for \( n = 2 \). Specifically, show that \( \Sigma_2 \subseteq \exists^2 \). More precisely, recall that \( \Sigma_2 = \{ L \mid \exists \text{TM} \ M^{\text{ATM}}(\cdot) \text{ s.t. } L(M^{\text{ATM}}) = L \} \). For each \( L \in \Sigma_2 \), you have to find a recursive relation \( R(x, y, z) \) such that \( x \in L \iff \exists y \forall z \ R(x, y, z) \), i.e.

\[
M^{\text{ATM}}(x) \text{ accepts } \iff \exists y \forall z \ R(x, y, z)
\]