Whenever calculations are needed to solve a problem, those calculations must be submitted as part of the homework assignment.

Homework must be submitted electronically, by 11:59pm on the due date. Unless express permission has been given in advance by the instructor for a late homework submission, a 30% percent penalty will be deducted for each late day (or part of a late day).

Exercise 1.1. (Sequences and rates of convergence.) Let $\gamma = 0.95$.

(a) Compute and print (using scientific notation, showing at least 12 digits) the first 11 elements of the sequence $\{x_k\}$, $k = 0, \ldots$, where $x_k = \gamma^{2^k}$. (Note that the exponent is $2^k$!) What is the limit of this sequence? What is the rate of convergence? Explain.

(b) Compute and print the first twelve elements of the sequence $\{x_k\}$, where $x_k = \gamma^{k^2}$. What is the limit of the sequence? Is the sequence converging superlinearly or not? Explain.

Exercise 1.2. Consider a nonlinear scalar-valued function $f(x)$ and assume that you are given two points $a_0$ and $b_0$, the endpoints of an interval such that $f(a_0)f(b_0) < 0$. Write a program that applies the bisection algorithm to $f$ in this interval. Your program should stop in one of the following three ways: (i) your code finds $x^*$ where $f(x^*) = 0$, (ii) the most recent interval of uncertainty is less than or equal to a tolerance $xtol$, or (iii) $\maxit$ bisection steps have been performed. (Suggested values for this problem: $xtol = 10^{-14}$ and $\maxit = 12$.)

At each bisection step, print the iteration index $k$, the endpoints of the $k$th interval of uncertainty $[a_k,b_k]$, and the values $f(a_k)$ and $f(b_k)$. Be sure to use scientific notation, and to show at least 12 digits in non-integer numbers.

(a) Consider the function $f(x) = x^3 + 2x - 3$. Show that a zero $x^*$ of $f(x)$ must exist in $[0.9,6]$. Is there more than one point $x^*$ where $f(x^*) = 0$? Explain.

(b) Run your bisection program on $f(x) = x^3 + 2x - 3$ with $a_0 = 0.9$, $b_0 = 6$, and $\maxit = 12$. Does the final interval of uncertainty contain $x^*$? Explain whether the results (e.g., the size of each interval of uncertainty, the pattern of convergence) are what you expected from the theoretical properties of bisection.

Exercise 1.3. Consider the function

$$\tilde{f}(x) = f = x^9 - 9x^8 + 36x^7 - 84x^6 + 126x^5 - 126x^4 + 84x^3 - 36x^2 + 9x - 1, \quad (1.1)$$

which is algebraically equivalent to $(x - 1)^9$. 
(a) Show mathematically that \( \tilde{f} \) has only one zero, at \( x^* = 1 \).

b) Let \( a_0 = 0.95 \), \( b_0 = 1.01 \), and \( \text{maxit} = 9 \). Run your bisection program for 10 iterations, evaluating \( \tilde{f} \) in exactly the form given in (1.1)—do not simplify the algebra! At each iteration, print \( k \), \( a_k \), \( b_k \), \( |a_k - b_k| \), and the associated values of \( \tilde{f} \) (using scientific notation for all non-integers, showing 9 digits of precision). What is the interval of uncertainty at the last bisection iteration? Explain whether or not the results are what you would expect and, if not, why. What do these results signify about the reliability of the bisection algorithm?

c) Run your bisection program a second time, this time applied to \( \tilde{f}(x) = (x - 1)^9 \), evaluated in this form, using the same \( a_0 \), \( b_0 \), and \( \text{maxit} \) as in part (b). Given that, mathematically, \( \tilde{f}(x) \equiv \tilde{f}(x) \), explain any differences in behavior between the iterates of (b) and (c).

Exercise 1.4. The function

\[
f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}
\]

has exactly one zero, at \( x^* = 1 \), in the interval \([0, 3]\). Let \( a_0 = 0 \) and \( b_0 = 3 \), and confirm that \( f(a_0)f(b_0) < 0 \). Then run your bisection program on \( f(x) \), with \([a_0, b_0]\) as an initial interval of uncertainty, stopping after 12 iterations or when the interval of uncertainty is less than or equal to 0.002. Does bisection appear to be converging to \( x^* \)? Explain, based on the numbers from your program, why your answer is “yes” or “no”. If “no”, explain how this behavior is consistent with the theory of bisection.

Exercise 1.5. (Special conditions for Newton’s method.) Consider using Newton’s method to solve for a zero of the function \( f(x) \), where \( f \) is a twice-continuously differentiable function for which we know \textit{a priori} that \( f''(x) > 0 \) for all \( x \). Suppose that we are given a finite interval \([a, b]\), and that \( f'(a) > 0 \) and \( f(a)f(b) < 0 \).

(a) Show that: (i) \( f(a) < 0 \); (ii) \( f'(x) \neq 0 \) for any \( x \in [a, b] \); and (iii) there exists only one zero of \( f(x) \) in \([a, b] \). [You may want to use one or more versions of the Mean Value Theorem and Rolle’s Theorem, given below.]

(b) Suppose that the point \( x_0 \) is in \([a, b]\) and that \( f(x_0) > 0 \). Show that, for every Newton iterate \( x_k \) with \( k \geq 1 \), (i) \( x_k < x_{k-1} \) and (ii) \( f(x_k) > 0 \).

(c) Consider the function \( \hat{f}(x) = x^2 - c \) for a constant \( c > 0 \). Given \( c > 0 \), using the operations of comparison, addition, multiplication, and division, describe an efficient (in terms of computation) way to choose \( a \) and \( b \) so that (i) \( \hat{f}(a) < 0 \) and (ii) \( \hat{f}(b) > 0 \). Note: There are many possible answers to this question.

(d) Pick a value \( c > 0 \) and a value of \( x_0 \) that is \textit{much larger} than \( \sqrt{c} \). Perform a single Newton iteration to solve \( \hat{f}(x) = 0 \), printing \( x_0 \) and \( x_1 \). What do you observe about the Newton iterate \( x_1 \) with respect to the interval of uncertainty \([0, x_0]\)? Explain.

(e) Consider applying Newton’s method to \( \tilde{f}(x) = x^2 - c \) with \( c = 0 \), starting with \( x_0 = 1 \). What sequence of iterates will be generated? What rate of convergence do they display? Explain.

Exercise 1.6. Suppose that \( f \) is a nonlinear scalar-valued twice-continuously differentiable function such that \( f''(x) > 0 \) for all \( x \). Assume that two points \( x_0 \) and \( x_1 \) are given such that
\( f(x_0)f(x_1) < 0 \), and that the regula falsi method is applied to find \( x^* \) such that \( f(x^*) = 0 \), using \( x_0 \) and \( x_1 \) to define an initial interval. Show that one of the initial points will be used in the linear fit in every regula falsi iteration.

**Exercise 1.7.** Write four programs (Newton, secant, regula falsi, Wheeler) that implement the associated methods for finding a zero of the nonlinear scalar-valued function \( f(x) \). The Newton code will need the values of \( f \) and \( f' \) at each iterate and is initialized with one point, \( x_0 \). The other three will need the value of \( f \) at each iterate, and will be initialized with two starting points, \( x_0 \) and \( x_1 \), such that \( f(x_0)f(x_1) < 0 \). (Note that this is not a requirement in general for the secant method.) Each program should stop after \( \text{maxit} \) iterations or if \( |f(x_k)| \leq \text{ftol} \) at the current iterate \( x_k \). At each iteration in each program, print \( k \), \( x_k \), and \( f(x_k) \).

(a) Write the code needed to evaluate the function \( f(x) = (x/2)^2 - 3\sin(2x) \) and its first derivative. Discuss any properties of this function that may affect the performance of methods for zero-finding. Study of a simple plot (using Matlab’s \texttt{plot} routine, for example) may be helpful in analyzing these properties.

(b) Run the Newton code with four different starting points: (i) \( x_0 = 0.8 \), (ii) \( x_0 = -2.3 \), (iii) \( x_0 = 1.2 \), and (iv) your choice of \( x_0 \), preferably selected to make the results “interesting”. In each case, comment on the behavior of the Newton iterates, explaining what happens to the Newton iterates and why. Also explain why the results with your chosen \( x_0 \) are especially interesting.

(c) Run the secant, regula falsi, and Wheeler codes with three pairs of initial points: (i) \( x_0 = 0.2 \) and \( x_1 = 3 \); (ii) \( x_0 = -3 \) and \( x_1 = 1.2 \); (iii) \( x_0 = -2 \) and \( x_1 = 3 \); and (iv) your choice of \( x_0 \) and \( x_1 \), preferably selected so that the results are interesting. Comment on the behavior of the iterates in each of the three methods, for each pair of starting points. Explain whether each method behaved as you expected compared to the others.

**Exercise 1.8.** Consider the linear equality constraints \( Ax = b \) and the vector \( \tilde{x} \):

\[
A = \begin{pmatrix}
3 & -4 & 6 & -1 & 7 \\
2 & 2 & -3 & 4 & -1
\end{pmatrix}, \quad b = \begin{pmatrix}
-40 \\
7
\end{pmatrix}, \quad \text{and} \quad \tilde{x} = (-3, 3, -2, 0, -1)^T.
\]

Let \( c = (-11, -4, 6, -15, -3)^T \) and \( d = (1, 1, 1, 1)^T \).

(a) Show (via computation) that \( \tilde{x} \) is feasible.

(b) Find any feasible point \( \bar{x} \) such that \( \bar{x} \neq \tilde{x} \) and explain how you found \( \bar{x} \).

(c) Consider the problem of minimizing the objective function \( \ell(x) = c^T x \) subject to \( Ax = b \).

(i) Is \( \tilde{x} \) optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether \( \tilde{x} \) is the unique minimizer, and whether or not the optimal value of the objective function is unique.

(iii) If \( \tilde{x} \) is not optimal, explain why it is not. In this case, find a direction \( p \) such that \( Ap = 0 \) and \( c^T p < 0 \) (and explain how you found \( p \)).

(d) Now consider the problem of minimizing \( \ell(x) = d^T x \) subject to \( Ax = b \).
(i) Is $\bar{x}$ optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether $\bar{x}$ is the unique minimizer, and whether or not the optimal value of the objective function is unique.

(iii) If $\bar{x}$ is not optimal, explain why it is not, and find a direction $p$ such that $Ap = 0$ and $d^T p < 0$. (Please explain how you found $p$.)

Exercise 1.9. Let $A$ be a nonzero $m \times n$ matrix, where $m > 0$ and $n > 0$. Assume that $a$ is an $n$-vector that is linearly independent of the rows of $A$. Let $e_{m+1}$ denote the $(m+1)$-th coordinate vector, and let $\bar{A}$ denote the $(m+1) \times n$ matrix

$$\bar{A} = \begin{pmatrix} A \\ a^T \end{pmatrix}.$$

Show that there must be a vector $p$ such that $\bar{A}p = e_{m+1}$, i.e., such that the equation $\bar{A}p = e_{m+1}$ is compatible.

Exercise 1.10. Consider a hyperplane $d^T x = \beta$, where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$, $d \neq 0$, and $\beta$ is a positive scalar. Find the $n$-vector $x^*$ of smallest two-norm that lies on the hyperplane, i.e. such that $\|x^*\|_2 \leq \|x\|_2$ among all $x$ satisfying $d^T x = \beta$. Explain how you found $x^*$ and show that it is optimal. What is $\|x^*\|_2$?

Exercise 1.11. Let $d$ denote a nonzero vector in $\mathbb{R}^n$ and let $\beta$ be a positive scalar. Consider the constraints $d^T x \geq \beta$ and $x \geq 0$, and assume that feasible points exist. Write down the solution to the linear program of minimizing $e^T x$ subject to these constraints, where $e = (1, 1, \ldots, 1)^T$ is the $n$-vector of all ones. Is the optimal point $x$ unique? Explain.

Exercise 1.12. Let $A$ be an $m \times n$ matrix.

(a) Show that if $b$ is an $m$-vector such that $b_i \leq 0$ for $i = 1, \ldots, m$, then at least one feasible point must exist for the combined constraints $Ax \geq b$ and $x \geq 0$. Is the result true for a general vector $b$? Explain why or why not.

(b) Consider the constraints $Ax \geq b$ and $x \geq 0$ for a general vector $b$, and assume that a feasible point exists. Must a vertex exist? Explain why or why not.

Exercise 1.13. Consider the linear program of minimizing $c^T x$ subject to $Ax \geq b$, where $A$ is $m \times n$. Assume that $x^*$ is a nondegenerate vertex and let $\hat{A}$ denote the active-constraint matrix at $x^*$. Assume that

$$c = (\hat{A})^T \hat{\lambda}$$

and $\hat{\lambda} \succeq 0$,

but $\hat{\lambda}_i = 0$ for at least one index $i$ (i.e. at least one component of $\hat{\lambda}$ is zero). Under these conditions, prove that $x^*$ is not the unique optimal solution of the linear program.

Theorem 0.1. (First mean value theorem.) If $\psi : D \subseteq \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is differentiable on $(a, b)$ and continuous on $[a, b]$, then there exists $\xi \in (a, b)$ such that

$$\psi(b) - \psi(a) = \psi'(\xi)(b - a).$$
Theorem 0.2. (Second mean value theorem.) Let $f : \mathcal{D} \subseteq \mathbb{R}^1 \to \mathbb{R}^1$ have a second derivative on the open interval $\mathcal{D}_0 \subseteq \mathcal{D}$. Then for any $x, x + p \in \mathcal{D}_0$, there is a point strictly between $x$ and $x + p$, i.e., a scalar $t$ satisfying $0 < t < 1$, such that

$$f(x + p) - f(x) - f'(x)p = \frac{1}{2}p^2 f''(x + tp),$$

where $f'$ and $f''$ are the first and second derivatives of $f$.

Theorem 0.3. (Rolle’s Theorem.) Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be differentiable in $(a, b)$ and assume that $f$ is continuous at both endpoints $a$ and $b$. If $f(a) = f(b)$, then there is at least one interior point $\xi$ between $a$ and $b$ such that $f'(\xi) = 0$. 