Exercise 2.1. A square matrix $A$ is symmetric if $a_{ij} = a_{ji}$.

(a) Find two specific symmetric square matrices $A$ and $B$ of dimension at least $2 \times 2$, with nonzero integer entries, such that the product $AB$ is not symmetric. Give $A$, $B$, and $AB$.

(b) Find two specific symmetric square matrices $A$ and $B$ of dimension at least $2 \times 2$, with nonzero integer entries, whose product $AB$ is symmetric. Give $A$, $B$, and $AB$.

(c) What condition is necessary in the $2 \times 2$ case for $AB$ to be symmetric when $A$ and $B$ are symmetric? Explain how you derived it.

Exercise 2.2. [Rank of a square triangular matrix.]

(a) Prove that a square upper-triangular matrix $R$ is singular only if at least one diagonal element is zero. (The same result is true for a lower-triangular matrix.)

(b) Assume that $R$ is an $n \times n$ triangular matrix, and that $k$ of its diagonal elements are zero. Must the rank of $R$ be exactly $n - k$? Explain your answer. If your answer is “yes”, give a proof. If your answer is “no”, give a counterexample.

Exercise 2.3. Consider the $n \times n$ elementary matrix $E = I - \alpha xy^T$.

(a) Show that $E$ is singular if and only if $\alpha x^T y = 1$.

(b) If $\alpha x^T y \neq 1$, show that $E^{-1} = I - \beta xy^T$, where $\beta = \alpha / (\alpha x^T y - 1)$.

Exercise 2.4. Let $x$ and $y$ be $n$-vectors, and let $Z$ be the rank-one matrix $Z = xy^T$. Describe an efficient way to compute $Z^k$ (the $k$-th power of $Z$) for $k > 1$. 
Exercise 2.5. Consider the following $A$ and $b$:

\[
A = \begin{pmatrix}
3 & 3 & 6 \\
2 & -3 & 6 \\
1 & 6 & 0 \\
2 & 0 & 1 \\
6 & 3 & 0
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
4 \\
3 \\
1 \\
-1 \\
-5
\end{pmatrix}.
\]

(a) What is the rank of $A$? How did you determine this value? What conclusion do you draw about whether the columns of $A$ are linearly independent (or not)? Explain.

(b) Is $b$ in the range of $A$? If your answer is “yes”, explain how you decided this. In this case, find the coefficients in the linear combination such that $b$ is a linear combination of the columns of $A$, and explain how you found them. If your answer is “no”, explain your reasoning.

(c) Let $r = \text{rank}(A)$. Find two subsets of $r$ linearly independent rows of $A$ (called “basic sets”), and explain how you determined that the designated rows were linearly independent.

(d) Use these two basic sets to solve the system $Ax = b$ for $x$, using a different basic set in each calculation of $x$.

(e) Is $x$ the same in both cases? Explain why or why not.

Exercise 2.6. [Properties of $A^T A$.]

(a) If the columns of $A$ are linearly independent, show that the matrix $A^T A$ is nonsingular.

(b) If $A$ has linearly dependent columns, show that $A^T A$ is singular.

Exercise 2.7. Here are the four standard properties of the norm $\|A\|$ of a real matrix $A$: (1) $\|A\| > 0$ if $A \neq 0$, and $\|0\| = 0$; (2) $\|\gamma A\| = |\gamma| \|A\|$ for any scalar $\gamma$; (3) $\|A + B\| \leq \|A\| + \|B\|$ for any $B$ of compatible dimensions; (4) $\|AB\| \leq \|A\| \|B\|$ for any $B$ of compatible dimensions.

(a) Show that the quantity $M = \max_{i,j} |a_{ij}|$, i.e., the largest absolute value of any element in the matrix $A$, satisfies properties (1), (2), and (3) of a matrix norm.

(b) Show that property (4) does not hold by giving a counterexample, i.e., specific matrices $A$ and $B$ such that property (4) is not satisfied.

Exercise 2.8. [Properties of triangular matrices.] Show that the product of two square upper-triangular matrices is upper triangular.

Exercise 2.9.

(a) Consider an $n \times n$ upper-triangular matrix $U$ such that

\[u_{11}u_{22} \cdots u_{n-1,n-1} \neq 0, \quad \text{but} \quad u_{nn} = 0,\]

i.e., $U$ is singular. Give a general algorithm for computing a nonzero vector $x$ such that $Ux = 0$.

(b) Verify your algorithm by finding a solution $x$ when $U$ is given by

\[
U = \begin{pmatrix}
1 & -2 & \frac{1}{2} \\
& 3 & 2 \\
& & 0
\end{pmatrix}.
\]

(c) What is the general form of $x$ satisfying $Ux = 0$ for this particular $U$?