Problem 6-1 (Preorder and Postorder Tree Walks) 8 Points

Consider a binary search tree. For a node $p$, denote by $p_{\text{pre}}$ and $p_{\text{post}}$ the positions of $p$ in the output sequences of PREORDER-TREE-WALK and POSTORDER-TREE-WALK, respectively.\(^1\)

(a) (2 points) Show that if node $q$ is in node $p$’s subtree, then $p_{\text{pre}} < q_{\text{pre}}$ and $p_{\text{post}} > q_{\text{post}}$.

(b) (6 points) Is the converse (i.e., if $p_{\text{pre}} < q_{\text{pre}}$ and $p_{\text{post}} > q_{\text{post}}$, then $q$ is in $p$’s subtree) true as well? If so, prove it; otherwise, provide a counter example.

Problem 6-2 (Depths in Randomly Built Search Trees) 14 Points

Consider the randomly built binary search trees (RBST) discussed in the lecture and denote by $D^{(1)}_n$ the depth of the element with rank 1 in an $n$-node RBST. Let $D(n) := \text{E}\left[D^{(1)}_n\right]$. Define $D(0) := 0$.

(a) (4 points) Determine $D(1)$, $D(2)$, and $D(3)$.

(Hint: For each $n = 1, 2, 3$, write down all possible binary search trees on $n$ nodes and compute their probabilities as well as the depth of the smallest element.)

(b) (6 points) Using the fact that the rank of the root of an RBST is a random integer $i$ from $\{1, \ldots, n\}$ and that the left subtree is an RBST on $i - 1$ nodes, derive a recurrence for $D(n)$ for general $n \geq 2$.\(^2\)

(c) (4 points) Prove that your recurrence solves to $D(n) = H_n - 1$, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$ is the $n^{\text{th}}$ harmonic number.

(Hint: For $n \geq 3$, consider the expression $nD(n) - (n - 1)D(n - 1)$.)

Problem 6-3 (Search in an Unsorted Tree) 14 Points

Consider a binary tree (BT) in which every node $v$ contains a unique key $v.key$ but in which the binary search tree property does not necessarily hold. In this task you will study the effects of using the usual search algorithm for BSTs in such a tree.

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\(^1\)Recall that PREORDER-TREE-WALK and POSTORDER-TREE-WALK print the root before resp. after the values in the subtrees.

\(^2\)Perhaps also have a look at the proof of Theorem 12.4 in CLRS for inspiration.
(a) (4 points) Consider the edge labeling that assigns 1 to an edge \((p, c)\), where \(p\) is the parent of \(c\), if “value in \(c\) > value in \(p\) and \(c\) is the right child of \(p\); or value in \(c\) < value in \(p\) and \(c\) is the left child of \(p\)” and 0 otherwise. Prove or disprove: \(\text{Tree-Search}(v.\text{key})\) finds \(v\) if and only if all edges from the root to \(v\) are labeled 1.

(b) (6 points) For every node \(v\) in a BT with real-numbered keys, let \(v.\text{low}, v.\text{high} \in \{-\infty, \infty\} \cup \mathbb{R}\) be such that \(\text{Tree-Search}(x)\) traverses \(v\) if and only if \(x \in (v.\text{low}, v.\text{high})\).\(^3\)

Describe in pseudo-code a linear-time algorithm based on pre-order tree traversal that assigns to each node \(v\) values \(v.\text{low}\) and \(v.\text{high}\) such that the condition above is satisfied. Argue why your algorithm is correct.

(c) (4 points) Explain how to extend your algorithm in (a) such that it assigns to each node \(v\) the value \(v.\text{found}\) indicating whether \(v\) is found if \(v.\text{key}\) is searched for using \(\text{Tree-Search}\).

**Problem 6-4 (Order Matters!) 14 (+4) points**

Assume we insert sequences of English letters into an empty 2-3-tree, following the standard alphabetical ordering when making comparisons: \(A < B < C < \ldots < Z\).

(a) (3 points) Assume you insert the letters \(R, E, L, A, T, I, O, N, S\) into a fresh 2-3-tree \(T\) in that exact order. Draw the resulting tree. What is its height?

(b) (4 points) Let \(h\) be your answer to part (a). Is it possible to get less than \(h\) by inserting the letters from (a) in a different order? If not, prove it. If yes, give the best ordering you can. What is the depth \(h_{\text{min}}\) you get?

(c) (3 points) Can you achieve \(h_{\text{min}}\) with lexicographic order \(A, E, I, L, N, O, R, S, T\)? Draw the resulting tree.

(d) (4 points) Extra credit: Can you achieve \(h_{\text{min}}\) for part (b) using an English word that is a permutation of \(R, E, L, A, T, I, O, N, S\)?

(e) (4 points) Give an example of a 2-3-tree \(T\) with exactly 5 leaves and two distinct values \(a, b\), such that one gets different final trees if one first inserts \(a\) into \(T\) and then deletes \(b\), instead of first deleting \(b\) from \(T\) and then inserting \(a\). Explain what happened.

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\(^3\)Note that if \(v.\text{low} \geq v.\text{high}\), then \((v.\text{low}, v.\text{high}) = \emptyset\).