CSCI-GA.1180:
Mathematical Techniques for Computer Science Applications
New York University, Fall 2015
Instructor: Margaret Wright

Homework Assignment 3
Assigned 8 October 2015; due 11:59pm, 20 October 2015

Unless stated otherwise, \( A \) is assumed to be a nonzero real \( m \times n \) matrix, where \( m \) may be different from \( n \), and \( m > 1 \) and \( n > 1 \). If a matrix-vector product or matrix-matrix product is mentioned, assume that the dimensions are compatible. Note that, in HW3–1 and HW3–2, you are asked about specific vector and matrix norms. Problems HW3–3 and HW–6 apply to any subordinate matrix norm. Problem HW3–7 explicitly uses the two-norm, which is the default in Matlab.

**HW3–1.** [Vector norms.]

(a) Show that, for any vector \( x \), it holds that \( \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \).

(b) Give a specific 3-vector \( z \) satisfying \( \|z\|_1 = \|z\|_2 = \|z\|_\infty \).

(c) Characterize all \( n \)-vectors \( x \) that satisfy \( \|x\|_1 = \|x\|_2 = \|x\|_\infty \).

**HW3–2.** [Matrix norms.] Consider the matrix

\[
A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.
\]

(a) Give the value of \( \|A\|_1 \) and the value of \( \|A\|_\infty \).

(b) Find a 2-vector \( u \) such that \( \|u\|_1 = 1 \) and \( \|Au\|_1 = \|A\|_1 \).

(c) Find a 2-vector vector \( v \) such that \( \|v\|_\infty = 1 \) and \( \|Av\|_\infty = \|A\|_\infty \).

**HW3–3.** Let \( \|\cdot\| \) be any subordinate matrix norm, as defined in the notes on norms (from the course website).

(a) Suppose that \( E \) is any square matrix. Show that, if \( I + E \) is singular, where \( I \) is an appropriately dimensioned identity matrix, then \( \|E\| \geq 1 \).

(b) Let \( F \) be any square matrix. Show that \( I - F \) is nonsingular if \( \|F\| < 1 \).
HW3–4. Let \( \tilde{x} \) be an approximate solution of \( Ax = b \), where \( A \) is a nonsingular \( n \times n \) matrix, \( n > 1 \), and \( b \) is a given vector. Let \( r \) denote the residual vector \( r = b - A\tilde{x} \), and define \( E \) as the rank-one matrix
\[
E = r \tilde{x}^T \frac{1}{\tilde{x}^T \tilde{x}}.
\]
Show that \( E \) satisfies \( (A + E)\tilde{x} = b \).

HW3–5. Consider an \( n \times n \) upper-triangular matrix \( U \) such that
\[
\begin{align*}
u_{11} & \neq 0, \\
u_{22} & \neq 0, \\
& \vdots \\
u_{n,n-1} & \neq 0,
\end{align*}
\]
(Thus, \( U \) is singular.)

(a) Give a “general” algorithm, meaning one that will work for any value of \( n \), for computing a nonzero \( n \)-vector \( x \) such that \( Ux = 0 \).

(b) What is the vector \( x \) produced by applying your algorithm to the matrix
\[
U = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 \\ 0 \end{pmatrix}?
\]

HW3–6. Given a nonsingular matrix \( A \) and a vector \( b \), let \( x \) denote the exact solution of \( Ax = b \), and let \( \| \cdot \| \) denote any subordinate norm. Consider a matrix \( A \) such that \( \|A\| = 1 \) and a vector \( b \) such that \( \|b\| = 10^{-6} \). If \( \|x\| = 1 \), what do we know about \( \text{cond}(A) \) measured in the given norm? Explain.

HW3–7. Note: numerical calculations will be needed to solve this problem. Use of the two-norm is assumed.

Let
\[
A = \begin{pmatrix} 0.550 & 0.423 \\ 0.473 & 0.364 \end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix} 0.8757 \\ 0.7533 \end{pmatrix}.
\]

(a) Show that the exact solution \( x^* \) to \( Ax = b \) is \( x^* = (0.9, 0.9)^T \).

(b) Give the solution \( \tilde{x} \) computed by executing the Matlab “backslash” command \( A \backslash b \) (or the equivalent in octave or SciPy). Compute and print the vectors (i) \( d = \tilde{x} - x^* \), (ii) \( r^* = b - Ax^* \), and (iii) \( \tilde{r} = b - A\tilde{x} \). Comment on the relative size of their norms.

(c) Let \( \hat{x} \) be a potential solution of \( Ax = b \), with residual \( \hat{r} = b - A\hat{x} \), and define \( E \) as the rank-one matrix
\[
E = \frac{1}{\hat{x}^T \hat{x}} \hat{r} \hat{x}^T.
\]
Show mathematically that the exact matrix \( E \) defined in (1) satisfies the relation
\[
(A + E)\hat{x} = b.
\]
(d) Consider the specific vector $\hat{x}$

$$\hat{x} = \begin{pmatrix} 40.9 \\ -51.1 \end{pmatrix}.$$ 

Would you say that $\hat{x}$ is close to $x^*$ from (a)? Would you say that $\hat{x}$ is close to $\tilde{x}$ from (b)? Explain.

(e) Given the vector $\hat{x}$ from (d), compute and print (i) its residual $\hat{r} = b - A\hat{x}$, (ii) the matrix $E$ from (1), and (iii) $\|E\|_2$. Would you describe $\|E\|_2$ as "small" or "large"? Explain.

(f) Compute the solution $\bar{x}$ to $(A + E)\bar{x} = b$, using the "\" command, and print $\bar{x}$.

(g) Print $\|\bar{x} - \hat{x}\|$. Based on this norm, is $\bar{x}$ "close to" $\hat{x}$? Explain why or why not.

(h) Do you agree with the statement that $\hat{x}$ is close to the exact solution of a system that is close to the original system? Explain why or why not.