1. Let FORMULA-SAT be the satisfiability problem for arbitrary boolean formulas. We know that FORMULA-SAT is NP-complete, since it is a generalization of 3SAT. However, for this exercise, you are to give a direct reduction from IS (independent set) to FORMULA-SAT. Recall that an instance $I$ of IS consists of an undirected graph $G = (V,E)$ and a bound $B$, and a solution $S$ is a set of distinct vertices $v_1,\ldots,v_B$ such that no two of these vertices is connected by an edge in $E$. Your task is to exhibit a boolean formula $\phi$ such that $I$ has an IS-solution $S$ solution if and only if $\phi$ has a satisfying assignment.

Suggested construction: define $\phi$ using variables $x_{ij}$, where $i = 1,\ldots,B$ and $j = 1,\ldots,n$. Intuitively, $x_{ij} = 1$ corresponds to $v_i = j$. However, you will have to include in $\phi$ constraints that capture the requirements that the $v_i$’s are distinct and that there are no edges between any of the $v_i$’s.

2. Consider the following problem M. An instance of M is a collection of sets $S_1,\ldots,S_m$ and a bound $B$. A solution is a set $T$ containing $B$ distinct items, such that (i) each item in $T$ belongs to some $S_i$, and (ii) each $S_i$ contains at most one item in $T$.

Show that M is NP-complete by giving a reduction IS $\rightarrow$ M.

3. Recall the knapsack problem (K), which we know is NP-complete. An instance of K consists of a sequence of positive integers $(a_1,\ldots,a_n,t)$. A solution is a subset $S$ of the set of indices $1,\ldots,n$ such that $\sum_{i \in S} a_i = t$.

Consider the following variant of the knapsack problem, called it HK. An instance of HK consists of a sequence of positive integers $(a_1,\ldots,a_n)$. A solution is a subset $S$ of the set of indices $1,\ldots,n$ such that $\sum_{i \in S} a_i = \sum_{j \notin S} a_j$.

The goal of this exercise is to show that HK is NP-complete by giving a reduction K $\rightarrow$ HK.

To this end, consider the following reduction: given an instance $I = (a_1,\ldots,a_n,t)$ of K, we map it to the instance $I' = (a_1,\ldots,a_n,s,2t)$ of HK, where $s := \sum_{i=1}^n a_i$.

Show that $I$ has a K-solution if and only if $I'$ has an HK-solution.

4. Consider bin packing problem (BP). Intuitively, the problem is to pack a collection of items of varying sizes into as few bins as possible, where each bin has capacity $c$. We can formally define this as a search problem as follows.

An instance of BP is a sequence of positive integers $(a_1,\ldots,a_n,c,B)$. A solution is a partition $S_1,\ldots,S_B$ of the set of indices $1,\ldots,n$ such that for each $k = 1,\ldots,B$, we have $\sum_{i \in S_k} a_i \leq c$.

Show that BP is NP-complete.

Hint: give a reduction HK $\rightarrow$ BP, where HK is defined as in the previous problem.

5. Suppose that you were given a polynomial-time algorithm that determines if a given 3CNF formula has a satisfying assignment (the output of this algorithm is true or false). Using this algorithm as a subroutine, design a polynomial-time algorithm that not only determines if a given 3CNF formula has a satisfying assignment, but also outputs a satisfying assignment if one exists.

No Honors problem this week.