Some background

**Theorem.** Let $G = (V, E)$ be an undirected graph. The following are equivalent.

1. $G$ is a tree
2. every pair of vertices in $G$ is connected by a single path, which is simple (no vertices repeat)
3. $G$ is connected, but removing any edge makes it unconnected
4. $G$ is acyclic, but adding any edge makes it cyclic
5. $G$ is connected and $|E| = |V| - 1$
6. $G$ is acyclic and $|E| = |V| - 1$
A generic MST algorithm

- Suppose $A \subseteq E$ is contained in some MST
  
  An edge $e \in E$ is called \textbf{safe for} $A$ if $A \cup \{e\}$ is also contained in some MST

- Generic MST algorithm:

  \[
  A \leftarrow \emptyset \\
  \text{repeat } |V| - 1 \text{ times:} \\
  \quad \text{find } e \in E \setminus A \text{ that is safe for } A \\
  \quad A \leftarrow A \cup \{e\}
  \]
Recognizing safe edges

- **Definition:**
  
  - A **cut** \( C \) is a partition \((S, V \setminus S)\), where \( \emptyset \subsetneq S \subsetneq V \)
  
  - An edge \( e \in E \) **crosses** a cut \( C = (S, V \setminus S) \) if one endpoint of \( e \) lies in \( S \), and the other lies in \( V \setminus S \)
  
  - A cut \( C \) **respects** \( A \subseteq E \) if no edge in \( A \) crosses \( C \)
Cut Lemma:

- Let $G = (V, E)$ be a connected, undirected graph with weights $w : E \rightarrow \mathbb{R}$
- Let $A \subseteq E$ be a subset of some MST
- Let $C$ be a cut that respects $A$
- Let $e \in E$ be an edge of smallest weight that crosses $C$
- Then: $e$ is safe for $A$
Proof:

• Let $T$ be an MST containing $A$
• If $e \in T$, we’re done, so assume $e \notin T$
• Goal: construct an MST $T' \supseteq A \cup \{e\}$
• Let $e = \{u, v\}$
• Consider the unique path $p$ from $u$ to $v$ in $T$ (which is simple)
• Since $e$ crosses the cut $C$, there must be some $e'$ along $p$ that crosses $C$
• Set $T' := (T \setminus \{e'\}) \cup \{e\}$
• Want to show: $T'$ is an MST that includes $A$
Proof (cont’d):

- \((V, T')\) is a tree
  \(|T'| = |V| - 1\) and \((V, T')\) is connected
- \(T' \supseteq A\)
  \(C\) respects \(A\), \(e'\) crosses \(C\) \(\Rightarrow\) \(e' \notin A\)
- \(T'\) is an MST
  Both \(e\) and \(e'\) cross \(C\) \(\Rightarrow\) \(w(e) \leq w(e')\)
- QED