Whenever calculations are needed to solve a problem, those calculations must be submitted as part of the homework assignment.

**Exercise 3.1.** Consider the linear equality constraints $Ax = b$ and the vector $\tilde{x}$:

$$A = \begin{pmatrix} 3 & -4 & 6 & -1 & 7 \\ 2 & 2 & -3 & 4 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -40 \\ 7 \end{pmatrix}, \quad \text{and} \quad \tilde{x} = (-3, 3, -2, 0, -1)^T.$$

Let $c = (-11, -4, 6, -15, -3)^T$ and $d = (1, 1, 1, 1)^T$.

(a) Show (via computation) that $\tilde{x}$ is feasible.

(b) Find any feasible point $\bar{x}$ such that $\bar{x} \neq \tilde{x}$ and explain how you found $\bar{x}$.

(c) Consider the problem of minimizing $\ell_c(x) = c^Tx$ subject to $Ax = b$.

(i) Is $\tilde{x}$ optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether $\tilde{x}$ is the unique minimizer, and whether or not the optimal value of $\ell_c$ is unique.

(iii) If $\tilde{x}$ is not optimal, explain why it is not. In this case, find a direction $p$ such that $A p = 0$ and $c^T p < 0$ (and explain how you found $p$).

(d) Now consider the problem of minimizing $\ell_d(x) = d^Tx$ subject to $Ax = b$.

(i) Is $\tilde{x}$ optimal for this problem?

(ii) If “yes”, explain why. In this case, also explain whether $\tilde{x}$ is the unique minimizer, and whether or not the optimal value of $\ell_d$ is unique.

(iii) If $\tilde{x}$ is not optimal, explain why it is not, and find a direction $p$ such that $A p = 0$ and $d^T p < 0$. (Please explain how you found $p$.)

**Exercise 3.2.** Let $A$ be a nonzero $m \times n$ matrix, where $m > 0$ and $n > 0$. Assume that $a$ is an $n$-vector that is linearly independent of the rows of $A$. Let $e_{m+1}$ denote the $(m+1)$-th coordinate vector, and let $\tilde{A}$ denote the $(m+1) \times n$ matrix

$$\tilde{A} = \begin{pmatrix} A \\ a^T \end{pmatrix}.$$ 

Show that there must be a vector $p$ such that $\tilde{A} p = e_{m+1}$, i.e., such that the equation $\tilde{A} p = e_{m+1}$ is compatible.

**Exercise 3.3.** Consider a hyperplane $d^T x = \beta$, where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$, $d \neq 0$, and $\beta$ is a positive scalar. Find the $n$-vector $x^*$ of smallest two-norm that lies on the hyperplane, i.e. such that $\|x^*\|_2 \leq \|x\|_2$ among all $x$ satisfying $d^T x = \beta$. Explain how you found $x^*$ and show that it is optimal. What is $\|x^*\|_2$?
Exercise 3.4. Let \( d \) denote a nonzero vector in \( \mathbb{R}^n \) and let \( \beta \) be a positive scalar. Consider the constraints \( d^T x \geq \beta \) and \( x \geq 0 \), and assume that feasible points exist. Write down the solution to the linear program of minimizing \( e^T x \) subject to these constraints, where \( e \) is the \( n \)-vector of all ones (so that \( e^T x \) is the sum of the \( n \) components of \( x \)). Is the optimal \( x \) unique? Explain.

Exercise 3.5. Let \( A \) be an \( m \times n \) matrix.
(a) Show that if \( b \) is an \( m \)-vector such that \( b_i \leq 0 \) for \( i = 1, \ldots, m \), then at least one feasible point must exist for the combined constraints \( Ax \geq b \) and \( x \geq 0 \). Is the result true for a general vector \( b \)? Explain why or why not.
(b) Consider the constraints \( Ax \geq b \) and \( x \geq 0 \) for a general vector \( b \), and assume that a feasible point exists. Must a vertex exist? Explain why or why not.

Exercise 3.6. Let \( A \) be a nonzero \( m \times n \) matrix and \( c \) an \( n \)-vector.
(a) If \( c = A^T \lambda \) for some \( \lambda \geq 0 \), show that \( c^T p \geq 0 \) for all \( p \) such that \( Ap \geq 0 \).
(b) If \( c \neq A^T \lambda \) for any \( \lambda \), show that there exists \( p \) such that \( c^T p < 0 \) and \( Ap \geq 0 \).

Exercise 3.7. Consider the linear program of minimizing \( c^T x \) subject to \( Ax \geq b \), where \( A \) is \( m \times n \). Assume that \( x^* \) is a nondegenerate vertex and let \( \bar{A} \) denote the active-constraint matrix at \( x^* \). Assuming that
\[
c = \bar{A}^T \bar{\lambda} \quad \text{and} \quad \bar{\lambda} \geq 0,
\]
but \( \bar{\lambda}_i = 0 \) for at least one index \( i \) (i.e. at least one component of \( \bar{\lambda} \) is zero), prove that \( x^* \) is not the unique solution of the linear program.

Exercise 3.8. Consider the two constraints \( x_1 - x_2 \geq 0 \) and \( x_1 + 2x_2 \leq 6 \), which intersect at \( \bar{x} = (2, 2)^T \). Suppose that we wish to add a third constraint, \( \alpha x_1 + x_2 \geq \gamma \), with \( \alpha \geq 0 \).
(a) Given a specific value of \( \alpha \), what must be the value of \( \gamma \) to ensure that the new constraint intersects the first two constraints at \( \bar{x} \)?
(b) With \( \gamma \) taken as the value determined in part (a), analyze the role of \( \alpha \) in the existence or nonexistence of feasible directions at \( \bar{x} \) with respect to all three constraints. Can you find \( \alpha_1 \geq 0 \) such that feasible directions with respect to all three constraints exist if \( \alpha \geq \alpha_1 \) but no such feasible directions exist if \( 0 \leq \alpha < \alpha_1 \)? Explain your answer.