1. Suppose a die is rolled repeatedly until it comes up ‘1’ or is rolled \( n \) times, whichever comes first. Let \( X \) be the random variable representing the number of rolls. Calculate \( E[X] \).
   Hint: use the fact that \( E[X] = \sum_{k \geq 1} \Pr[X \geq k] \)

2. Let \( X \) be a random variable that takes only nonnegative integer values. Show that \( E[X^2] = \sum_{k \geq 1} (2k - 1) \Pr[X \geq k] \)
   Hint: write \( X = \sum_{k \geq 1} X_k \), where \( X_k \) is the indicator variable for the event \( X \geq k \).

3. Let \( S \) be a randomly chosen subset of \( \{1, \ldots, n\} \). That is, for each \( i = 1, \ldots, n \), we include \( i \) in \( S \) with probability \( 1/2 \).
   Let \( X = |S| \). Calculate \( E[X] \) and \( E[X^2] \).
   Hint: write \( X = \sum_{i=1}^n X_i \), where \( X_i \) is the indicator variable for the event \( i \in S \).

4. Consider the following recursive, probabilistic algorithm \( A \), which takes as input a finite set \( S \) of items.

   Algorithm \( A(S) \):
   
   if \( |S| \leq 1 \) return \((0, 0, 0)\)
   else
     let \( R \) be a randomly chosen subset of \( S \)
     \((v_1, v_2, v_3) \leftarrow A(R)\)
     \((w_1, w_2, w_3) \leftarrow A(S \setminus R)\)
     return \((\max\{v_1, w_1\} + 1, v_2 + w_2 + |S|, v_3 + w_3 + |S|^2)\)

Let \((X_1, X_2, X_3)\) denote the output of \( A \) on input \( S \), and let \( n := |S| \). Show that \( E[X_1] = O(\log n) \), \( E[X_2] = O(n \log n) \), and \( E[X_3] = O(n^2) \).

5. In a variant of quick sort, instead of choosing the pivot element at random, we pick a “better” pivot using the following strategy: sample 3 elements of the array to be sorted at random, and select the median of these 3 elements as the pivot. More precisely, if the array is \( A[1, \ldots, n] \), then we choose \( i_1, i_2, i_3 \in \{1, \ldots, n\} \) uniformly and independently at random, sort the elements \( A[i_1], A[i_2], A[i_3] \), and take the element that comes second in sorted order.

   Intuitively, this “median of 3” variant should give better splits — splits more likely to be closer to \( n/2 \) in size. In this exercise, you are to analyze the behavior of this variant. For simplicity, assume that all elements of \( A \) are distinct; in fact, without loss of generality, just assume that the elements of \( A \) are the numbers \( 1, \ldots, n \), in some arbitrary order.

   (a) If \( M \) denotes the size of the subset of elements that is smaller than the pivot (computed using the median-of-3 strategy), show that \( E[M^2] \leq (6/20 + o(1))n^2 \).

   Hint: Let \( X_1, X_2, X_3 \) be random variables denoting the values \( A[i_1], A[i_2], A[i_3] \) above. For \( k = 1, \ldots, n \), estimate
   
   \[ p_k := \Pr[X_1 = k \text{ and } X_2 \leq k \text{ and } X_3 \geq k], \]

   and argue that \( \Pr[M = k - 1] \leq 6p_k \)

   (b) Based on part (a), derive an estimate for the expected depth of the recursion tree, using an argument analogous to the one presented in class for quick sort. How does your expected depth estimate compare to that of the original merge sort algorithm?