Plan: build up a catalog of \textbf{NP}-complete problems

Starting point — a special problem called \textit{3SAT}:

• We will show that a variety of problems are \textbf{NP}-complete, via sequences of reductions that ultimately end at \textit{3SAT}

• For the time being, we shall \textit{assume} that \textit{3SAT} is \textbf{NP}-complete

Building the catalog:

• To prove that a language \textit{L} is \textbf{NP}-complete:
  ○ Prove that \textit{L} ∈ \textbf{NP}
  ○ Prove that \textit{L} is \textbf{NP}-hard by showing that \textit{L′} \leq_{\text{P}} \textit{L} for some known \textbf{NP}-complete language \textit{L′}
Satisfiability (SAT)

Instance:

- A Boolean formula $\phi$:
  - variables $x_1, \ldots, x_m$
  - constants 0, 1
  - operators $\lor, \land, \neg$
  - Parentheses

Question:

- Is there an assignment to the variables $x_1, \ldots, x_m$ such that $\phi(x_1, \ldots, x_m) = 1$?

$SAT \in \text{NP}$:

- a satisfying assignment is a witness
3SAT: a special case of SAT

Conjunctive Normal Form:

- a conjunction (\(\land\)) of clauses
- each clause is a disjunction (\(\lor\)) of literals
- each literal is a variable \(x\) or its complement \(\bar{x}\)

Examples:

\[ x \land y, \quad \bar{x} \land (y \lor z), \quad (x \lor y \lor \bar{z}) \land (w \lor \bar{x} \lor z) \]

A special form: 3-CNF

- Each clause consists of 3 distinct literals
- \(3SAT := \{ \langle \phi \rangle : \phi \text{ is a satisfiable 3-CNF formula} \}\)
**CLIQUE: An NP-complete graph problem**

**Definition:**
- Let $G = (V, E)$ be an undirected graph
- A *clique* is a set $C \subseteq V$ such that $(u, v) \in E$ for all $u, v \in C$ such that $u \neq v$

**The CLIQUE problem:**
- **Instance:**
  - A pair $(G, k)$, where $G$ is an undirected graph and $k$ is a positive integer
- **Question:**
  - Is there a clique in $G$ of size $\geq k$?
Proof that \textit{CLIQUE} is \textbf{NP}-complete

- \textit{CLIQUE} $\in$ \textbf{NP}: clear, as the clique itself is a witness
- Need to show \textit{CLIQUE} is \textbf{NP}-hard
- Reduction: 3\textit{SAT} $\leq_P$ \textit{CLIQUE}
- Let $\phi$ be a 3-CNF formula:
  \[ \phi = (a_1 \lor b_1 \lor c_1) \land \cdots \land (a_k \lor b_k \lor c_k) \]
- Goal: construct a graph $G$ such that
  $\phi$ is satisfiable $\iff$ $G$ has a clique of size $k$
Proof (cont’d):

- $G$ has a $3k$ vertices, one for each literal
- There is an edge between two vertices unless
  1. the corresponding literals belong to the same clause, or
  2. the corresponding literals are contradictory (i.e., $x$ and $\overline{x}$)

Example:
\[
\phi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x}_2 \lor x_3) \land (\overline{x}_1 \lor \overline{x}_2 \lor x_3)
\]
Proof (cont’d):

• Need to show
  \[ \phi \text{ is satisfiable } \iff G \text{ has a clique of size } k \]

• \(\implies\):
  
  ◦ suppose \(\phi\) is satisfiable
  ◦ choose a satisfying assignment
  ◦ for each clause, pick one true literal
  ◦ this gives a \(k\)-clique (verify: every pair of vertices is connected)
Proof (cont’d):

- \( \Leftarrow:\)
  - suppose \( G \) has a \( k \)-clique \( C \)
  - by Rule 1, each triple can have at most one element in \( C \)
  - so \( C \) has exactly one element from each triple
  - by Rule 2, we can make a corresponding truth assignment that satisfies \( \phi \)