Social Network Structure

$S$ – Strong Ties, $W$ – weak ties
Levels: $L = 4$. Index $l = 1,2,3,4$. Proximity to Innovators.

Later, reorganize as: Innovators, Early Adopters, Late Adopters, Laggards

Groups per level: $K(l)$. Example $K(l = 1) = 1, K(l = 2) = 4, K(l = 3) = 8, K(l = 4) = 12$. Index $k$.

Nodes per group: $N_{l,k}$. Example $N_{l=2,k=3} = 4$. Index $i$ or $j$.

$$T(l,k,i) = S(l,k,i) + W(l,k,i), \text{ total number of ties by "ego"-}(l,k,i)$$
Social Network Structure

Each node is completely specified by a triple \((l,k,i)\), but let us focus on one clique \((l,k)\). In this clique, a node is described by index \(i\).

We have now to specify the edges on the graph.

**Strong Ties (bi-directed):** for each \((l,k)\) the nodes will be connected by strong ties, i.e.,

\[ e_S(j,i) \quad \forall i, j = 1, ..., N(l,k) \]

Assignment possibilities

a. The further away from innovation \(\rightarrow\)the stronger are the ties. People may only want to join an innovation if more of others will do so (threshold model).

b. Randomly assign different values.

**Weak Ties (Non symmetric bridges):**

\[ e_W(j',i) \]

but not all connections exist.

Note that this is a directed graph, so the innovation is coming from

\[ j' \rightarrow i \]

They occur more often from people at a level closer to the innovation towards people further away from innovation, i.e., \(l' \leq l\).
Social Network Structure

States and time \( s_i(t) = 0,1 \)

Probability of \( i \) to be at state \( s \) at time \( t \) : \( p_i^t(s, (t)) \)

How does Weak ties and Strong ties change \( p_i^t(s, (t)) \) over time?

**Probabilities**

\[
P(x) \geq 0 \quad P(x, y) \text{- joint distribution on variables } x \text{ and } y
\]

\[
\sum_x P(x) = 1 \quad (1)
\]

\[
P(x) = \sum_y P(x, y) \text{- marginal distribution } \left( P(y) = \sum_x P(x, y) \right) \quad (2)
\]

\[
P(x \mid y) = \frac{P(x, y)}{P(y)} \text{- conditional probability} \quad (3)
\]

**Example: two bias coins**

\( c_1, c_2 = H, T \) - coins can be head or tail (1 or 0).

\[
P(c_1 = H, c_2 = H) = p_{11}; P(c_1 = H, c_2 = T) = p_{10}; P(c_1 = T, c_2 = H) = p_{01}; P(c_1 = T, c_2 = T) = p_{00}
\]

(1) \( \rightarrow 1 = p_{11} + p_{10} + p_{01} + p_{00} \)

(2) \( \rightarrow P(c_1 = H) = p_{11} + p_{10}; P(c_1 = T) = p_{01} + p_{00}; P(c_2 = H) = p_{11} + p_{01}; P(c_2 = T) = p_{10} + p_{00} \)

(3) \( \rightarrow P(c_1 = H \mid c_2 = H) = \frac{p_{11}}{p_{11} + p_{01}}; P(c_1 = T \mid c_2 = H) = \frac{p_{01}}{p_{11} + p_{01}}; \)

\[
P(c_1 = H \mid c_2 = T) = \frac{p_{10}}{p_{10} + p_{00}}; P(c_1 = T \mid c_2 = T) = \frac{p_{00}}{p_{10} + p_{00}}
\]
Social Network Structure

**Strong Ties:** Two states that are in a strong tie relation, may likely adopt the same state. A way to express the influence is via the conditional probabilities

\[ p_{ij}^S(s_i, t) \mid s_j, (t-1) \rightarrow \begin{cases} 1 = p_{ij}^S(s_i, t = 1 \mid s_j, (t-1) = 1) + p_{ij}^S(s_i, t = 0 \mid s_j, (t-1) = 1) \\ 1 = p_{ij}^S(s_i, t = 1 \mid s_j, (t-1) = 0) + p_{ij}^S(s_i, t = 0 \mid s_j, (t-1) = 0) \end{cases} \]

There are two free parameters:

\[ a_{ij} = p_{ij}^S(s_i, t = 1 \mid s_j, (t-1) = 1) \quad \text{and} \quad b_{ij} = p_{ij}^S(s_i, t = 0 \mid s_j, (t-1) = 0) \]

Which specify how likely is a node \( j \) at time \( t-1 \) to influence \( i \) at time \( t \). Note that one can even influence negatively by having these numbers to be below 0.5.

It is clear from (1) that

\[ p_{ij}^S(s_i, t = 0 \mid s_j, (t-1) = 1) = 1 - a_{ij} \quad \text{and} \quad p_{ij}^S(s_i, t = 1 \mid s_j, (t-1) = 0) = 1 - b_{ij} \]

\[ p_{ij}^S(s_i, t) \mid s_j, (t-1) = \begin{pmatrix} p_{ij}^S(s_i = 1 \mid s_j = 1) & p_{ij}^S(s_i = 1 \mid s_j = 0) \\ p_{ij}^S(s_i = 0 \mid s_j = 1) & p_{ij}^S(s_i = 0 \mid s_j = 0) \end{pmatrix} = \begin{pmatrix} a_{ij} & 1 - b_{ij} \\ 1 - a_{ij} & b_{ij} \end{pmatrix} \]

Note that for each two nodes we have two directional edges and they can have different values, i.e., \( a_{ij} \neq a_{ji} \).
Evolution of node $i$ due to one Strong Tie from node $j$

$$p_{ij}^{t}(s_{i}(t)) = \sum_{s_{j}(t-1)=0}^{1} p_{ij}^{t,s}(s_{i}(t), s_{j}(t-1)) = \sum_{s_{j}(t-1)=0}^{1} p_{ij}^{s}(s_{i}(t) | s_{j}(t-1)) p_{j}^{t-1}(s_{j}(t-1))$$

In matrix –vector notation

$$\begin{pmatrix} p_{ij}^{t,s}(s_{i} = 1) \\ p_{ij}^{t,s}(s_{i} = 0) \end{pmatrix} = \begin{pmatrix} p_{ij}^{s}(s_{i} = 1 | s_{j} = 1) & p_{ij}^{s}(s_{i} = 1 | s_{j} = 0) \\ p_{ij}^{s}(s_{i} = 0 | s_{j} = 1) & p_{ij}^{s}(s_{i} = 0 | s_{j} = 0) \end{pmatrix} \begin{pmatrix} p_{j}^{t-1}(s_{j} = 1) \\ p_{j}^{t-1}(s_{j} = 0) \end{pmatrix}$$

$$= \begin{pmatrix} a_{ij} & 1-b_{ij} \\ 1-a_{ij} & b_{ij} \end{pmatrix} \begin{pmatrix} p_{j}^{t-1}(s_{j} = 1) \\ p_{j}^{t-1}(s_{j} = 0) \end{pmatrix}$$

$$= \begin{pmatrix} a_{ij} p_{j}^{t-1}(s_{j} = 1) + (1-b_{ij}) p_{j}^{t-1}(s_{j} = 0) \\ (1-a_{ij}) p_{j}^{t-1}(s_{j} = 1) + b_{ij} p_{j}^{t-1}(s_{j} = 0) \end{pmatrix}$$

Note that we can remove the index of $s$ as it is the same as the index of $p$

**Special case: $j=i$.**

$$\begin{pmatrix} p_{ij}^{t,s}(s_{i} = 1) \\ p_{ij}^{t,s}(s_{i} = 0) \end{pmatrix} = \begin{pmatrix} a_{ii} & 1-b_{ii} \\ 1-a_{ii} & b_{ii} \end{pmatrix} \begin{pmatrix} p_{i}^{t-1}(s_{i} = 1) \\ p_{i}^{t-1}(s_{i} = 0) \end{pmatrix}$$

The larger is $b_{ii}$ and $a_{ii}$ the more inertia to new ideas
Social Network Evolution of node $i$ in a clique

Mixture of $N$ experts:

product: $p_i^t(s_i(t)) = \frac{1}{C_i} \prod_{j=1}^{N} p_{ij}^{t,s}(s_i(t))$,

average $p_i^t(s_i(t)) = \frac{1}{C_i} \sum_{j=1}^{N} p_{ij}^{t,s}(s_i(t))$,

geometric mean: $p_i^t(s_i(t)) = \frac{1}{C_i} \prod_{j=1}^{N} \sqrt[p_{ij}^{t,s}](s_i(t))$

Mixture of $N$ experts, products:

$p_i^t(s_i(t)) = \frac{1}{C_i} \prod_{j=1}^{N} p_{ij}^t(s_i(t)) = \frac{1}{C_i} \prod_{j=1}^{N} \left( \sum_{s_j(t-1)=0}^{1} p_{ij}^s(s_i(t) | s_j(t-1)) p_j^{t-1}(s_j(t-1)) \right) \text{ or }$

\[
\begin{pmatrix}
  p_i^t(s_i = 1) \\
  p_i^t(s_i = 0)
\end{pmatrix} = \begin{pmatrix}
  \prod_{j=1}^{N} p_{ij}^{t,s}(s_i = 1) \\
  \prod_{j=1}^{N} p_{ij}^{t,s}(s_i = 0)
\end{pmatrix} = \begin{pmatrix}
  \prod_{j=1}^{N} \left( a_{ii} \cdot p_j^{t-1}(s_j = 1) + (1 - b_{ii}) p_j^{t-1}(s_j = 0) \right) \\
  \prod_{j=1}^{N} \left( (1 - a_{ii}) \cdot p_j^{t-1}(s_j = 1) + b_{ii} \cdot p_j^{t-1}(s_j = 0) \right)
\end{pmatrix}
\]
Social Network Structure

**Weak Interactions among nodes (Weak Ties):** Weak ties are only one directional; one node brings the innovation to another node. We models as

\[ p_{ij}^W(s_i(t) = 1 | s_{j'}(t-1) = 1) = \alpha_{ij'} \]
\[ p_{ij}^W(s_i(t) = 0 | s_{j'}(t-1) = 0) = \beta_{ij'} \]

Normalizing the probabilities

\[ p_{ij}^W(s_i(t) = 0 | s_{j'}(t-1) = 1) = 1 - \alpha_{j'i}, \quad p_{ij}^W(s_i(t) = 1 | s_{j'}(t-1) = 0) = 1 - \beta_{j'i}; \]

we conclude

\[ p_{ij'}^W(s_i(t) | s_{j'}(t-1)) = \begin{pmatrix} \alpha_{ij'} & 1 - \beta_{ij'} \\ 1 - \alpha_{ij'} & \beta_{ij'} \end{pmatrix} \]

The larger is \( \alpha_{ij'}, \beta_{ij'} \), the more the nodes will tend to be on the same state, i.e., the more an innovation will be accepted.
Evolution of node $i$ due to a Weak Tie from $j'$

$$p_{ij}'(s_i(t)) = \left( \sum_{s_{j'}(t-1)=0}^{1} p_{ij}'(s_i(t) \mid s_{j'}(t-1)) \ p_{j'i}'(s_{j'}(t-1)) \right)$$

$$= \begin{pmatrix}
    p_{ij}'(s_i = 1) \\
    p_{ij}'(s_i = 0)
\end{pmatrix} = \begin{pmatrix}
    \alpha_{j'i} & 1 - \beta_{j'i} \\
    1 - \alpha_{j'i} & \beta_{j'i}
\end{pmatrix} \begin{pmatrix}
    p_{j'i}'(s_{j'} = 1) \\
    p_{j'i}'(s_{j'} = 0)
\end{pmatrix}$$

Note that we can remove the index of $s$ as it is the same as the index of $p$

Let us now take into account the inertia of the node $i$, its tendency to stay at the same state as it was before at $(t-1)$.

$$p_{ii}'(s_i(t)) = \left( \sum_{s_i(t-1)=0}^{1} p_{ii}'(s_i(t) \mid s_i(t-1)) \ p_i(t-1)(s_i(t-1)) \right)$$

$$= \begin{pmatrix}
    p_{ii}'(s_i = 1) \\
    p_{ii}'(s_i = 0)
\end{pmatrix} = \begin{pmatrix}
    a_{ii} & 1 - b_{ii} \\
    1 - a_{ii} & b_{ii}
\end{pmatrix} \begin{pmatrix}
    p_i(t-1)(s_i = 1) \\
    p_i(t-1)(s_i = 0)
\end{pmatrix}$$

How to combine both? ...
Evolution of node $i$ due to a Weak Tie from $j'$

Mixture of 2 experts

Expert 1-

$$
p_{ij'}^{t,W}(s_i(t)) = \begin{pmatrix} p_{ij'}^{t,W}(s_i = 1) \\ p_{ij'}^{t,W}(s_i = 0) \end{pmatrix} = \begin{pmatrix} \alpha_{j'i} & 1 - \beta_{j'i} \\ 1 - \alpha_{j'i} & \beta_{j'i} \end{pmatrix} \begin{pmatrix} p_{j'}^{t-1}(s_{j'} = 1) \\ p_{j'}^{t-1}(s_{j'} = 0) \end{pmatrix}
$$

Expert 2-

$$
p_{ii}^{t,W}(s_i(t)) = \begin{pmatrix} p_{ii}^{t,W}(s_i = 1) \\ p_{ii}^{t,W}(s_i = 0) \end{pmatrix} = \begin{pmatrix} a_{ii} & 1 - b_{ii} \\ 1 - a_{ii} & b_{ii} \end{pmatrix} \begin{pmatrix} p_{i}^{t-1}(s_i = 1) \\ p_{i}^{t-1}(s_i = 0) \end{pmatrix}
$$

Product:

$$
\begin{align*}
\left( p_i^t(s_i = 1) \right) & = \frac{1}{C_i} \left( p_{ij'}^{t,W}(s_i = 1) \cdot p_{ii}^{t,S/W}(s_i = 1) \right) \\
\left( p_i^t(s_i = 0) \right) & = \frac{1}{C_i} \left( p_{ij'}^{t,W}(s_i = 0) \cdot p_{ii}^{t,S/W}(s_i = 0) \right)
\end{align*}
$$

Geometric Mean

$$
\begin{align*}
\left( p_i^t(s_i = 1) \right) & = \frac{1}{C_i} \left( \sqrt{p_{ij'}^{t}(s_i = 1) \times p_{ii}^{t}(s_i = 1)} \right) \\
\left( p_i^t(s_i = 0) \right) & = \frac{1}{C_i} \left( \sqrt{p_{ij'}^{t}(s_i = 0) \times p_{ii}^{t}(s_i = 0)} \right)
\end{align*}
$$

Average:

$$
\begin{align*}
\left( p_i^t(s_i = 1) \right) & = \frac{1}{2} \left( p_{ij'}^{t,W}(s_i = 1) + p_{ii}^{t,S/W}(s_i = 1) \right) \\
\left( p_i^t(s_i = 0) \right) & = \frac{1}{2} \left( p_{ij'}^{t,W}(s_i = 0) + p_{ii}^{t,S/W}(s_i = 0) \right)
\end{align*}
$$