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- Geometry Theorem Proving
Outline

- Combining Decision Procedures

Sources:


Combining Decision Procedures

Often, verification conditions are expressed in a language which mixes several theories.

A natural question is whether one can use decision procedures for individual theories to construct a decision procedure for the union theory.

More precisely, suppose that $\Sigma_1, \ldots, \Sigma_n$ are $n$ signatures, and for $i = 1, \ldots, n$, let $T_i$ be a $\Sigma_i$-theory.

Then, let $\text{Sat}_i$ be a decision procedure for deciding the $T_i$-satisfiability of $\Sigma_i$-formulas.

How can we use these to construct a decision procedure for the $T$-satisfiability of $\Sigma$-formulas, where $T = \bigcup T_i$ and $\Sigma = \bigcup \Sigma_i$?

Let’s start by identifying some interesting theories that we may want to combine.
The Theory $T_{\mathcal{E}}$ of Equality

Consider the signature $(\mathbin{=}, f, g, \ldots, p, q, \ldots)$ with equality and some number of function and predicate symbols.

Given such a signature, the theory $T_{\mathcal{E}}$ is the theory $Cn \emptyset$.

The theory does not restrict the possible values of the symbols in its signature in any way. For this reason, it is sometimes called the theory of *equality with uninterpreted functions (EUF)*.

The satisfiability problem for $T_{\mathcal{E}}$ is just the satisfiability problem for first-order logic, which is undecidable.

The satisfiability problem for conjunctions of literals in $T_{\mathcal{E}}$ is decidable in polynomial time using *congruence closure*.  


The Theory $T_Z$ of Integers

Let $\Sigma_Z$ be the signature $(0, 1, +, −, \leq)$.

Let $A_Z$ be the standard model of the integers with domain $\mathbb{Z}$.

Then $T_Z$ is defined to be the set of all $\Sigma_Z$-sentences true in the model $A_Z$.

As showed by Presburger in 1929, the general satisfiability problem for $T_Z$ is decidable, but its complexity is super-exponential.

The quantifier-free satisfiability problem for conjunctions of literals in $T_Z$ is “only” NP-complete.

Why?
The Theory $T_Z$ of Integers

Let $\Sigma^\times_Z$ be the same as $\Sigma_Z$ with the addition of the symbol $\times$ for multiplication, and define $A_Z^\times$ and $T_Z^\times$ in the obvious way.

The satisfiability problem for $T_Z^\times$ is undecidable (a result shown as part of Gödel’s incompleteness theorem).

The question of satisfiability for quantifier-free formulas in $T_Z^\times$ is equivalent to Hilbert’s tenth problem. It was shown to be undecidable by Matiyasevich in 1971.
The Theory $T_{\mathcal{R}}$ of Reals

Let $\Sigma_{\mathcal{R}}$ be the signature $(0, 1, +, -, \leq)$.

Let $A_{\mathcal{R}}$ be the standard model of the reals with domain $\mathcal{R}$.

Then $T_{\mathcal{R}}$ is defined to be the set of all $\Sigma_{\mathcal{R}}$-sentences true in the model $A_{\mathcal{R}}$.

The satisfiability problem for $T_{\mathcal{R}}$ is decidable, but the complexity is doubly-exponential.

The quantifier-free satisfiability problem for conjunctions of literals in $T_{\mathcal{R}}$ is solvable in polynomial time, though exponential methods (like Simplex or Fourier-Motzkin) perform well in practice.
The Theory $T_R$ of Reals

Let $\Sigma_R^\times$ be the same as $\Sigma_R$ with the addition of the symbol $\times$ for multiplication, and define $A_R^\times$ and $T_R^\times$ in the obvious way.

In contrast to the theory of integers, the satisfiability problem for $T_R^\times$ is decidable though the complexity is inherently doubly-exponential.
The Theory $T_A$ of Arrays

Let $\Sigma_A$ be the signature $(\text{read}, \text{write})$.

Let $\Lambda_A$ be the following axioms:

$$
\forall a \forall i \forall v \left( \text{read}(\text{write}(a, i, v), i) = v \right)
$$

$$
\forall a \forall i \forall j \forall v \left( i \neq j \rightarrow \text{read}(\text{write}(a, i, v), j) = \text{read}(a, j) \right)
$$

$$
\forall a \forall b \left( (\forall i \left( \text{read}(a, i) = \text{read}(b, i) \right)) \rightarrow a = b \right)
$$

Then $T_A = Cn \Lambda_A$.

The satisfiability problem for $T_A$ is undecidable, but the quantifier-free satisfiability problem for conjunctions of literals in $T_A$ is decidable (the problem is NP-complete).
Theories of Inductive Data Types

An *inductive data type* (IDT) defines one or more *constructors*, and possibly also *selectors* and *testers*.

**Example: list of int**

- Constructors: $\text{cons}: (\text{int}, \text{list}) \rightarrow \text{list}$, $\text{null}: \text{list}$
- Selectors: $\text{car}: \text{list} \rightarrow \text{int}$, $\text{cdr}: \text{list} \rightarrow \text{list}$
- Testers: $\text{is_cons}$, $\text{is_null}$

The *first order theory* of a recursive data type associates a function symbol with each constructor and selector and a predicate symbol with each tester.

**Example:** $\forall x : \text{list}. \ (x = \text{null} \lor \exists y : \text{int}, z : \text{list}. \ x = \text{cons}(y, z))$
Theories of Inductive Data Types

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- Selectors: \(\text{car: list} \rightarrow \text{int, cdr: list} \rightarrow \text{list}\)
- Testers: \(\text{is_cons, is_null}\)

The *first order theory* of a recursive data type associates a function symbol with each constructor and selector and a predicate symbol with each tester.

**Example:** \(\forall x: \text{list}. (x = \text{null} \lor \exists y: \text{int, z: list}. x = \text{cons}(y, z))\)

For IDTs with a single constructor, a conjunction of literals is decidable in polynomial time using an algorithm by Oppen.

For more general IDTs, the problem is NP-complete, but reasonably efficient algorithms exist in practice.
The Theory $T_B$ of Fixed-Width Bitvectors

A natural domain for high-level reasoning about circuits and programs is the domain of bitvectors.

The theory $T_B$ allows variables to be declared as vectors of bits of any fixed length.

The function and predicate symbols in the theory include extraction, concatenation, bitwise Boolean operations, and arithmetic operations.

It is easy to see that the decision problem in $T_B$ is NP-complete by a simple reduction to SAT.

The challenge for this theory is to come up with more efficient solvers than the naïve conversion to SAT.
The Nelson-Oppen Method

A very general method for combining decision procedures is the \textit{Nelson-Oppen} method.

This method is applicable when

1. The signatures $\Sigma_i$ are disjoint.

2. The theories $T_i$ are stably-infinite.
   
   A $\Sigma$-theory $T$ is \textit{stably-infinite} if every $T$-satisfiable quantifier-free $\Sigma$-formula is satisfiable in an infinite model.

3. The formulas to be tested for satisfiability are quantifier-free.

In practice, only the third requirement is a significant restriction.
The Nelson-Oppen Method

The Nelson-Oppen method restricts its attention to conjunctions of literals.

We’ll talk about the more general case later.

Before presenting the method, we need a couple of definitions:

- A formula or term is \textit{i-pure} if it only contains symbols from signature $\Sigma_i$.

- If $S$ is a set of terms and $\sim$ is an equivalence relation on $S$, then the \textit{arrangement of $S$ induced by $\sim$} is $\{x = y \mid x \sim y\} \cup \{x \neq y \mid x \not\sim y\}$. 
The Nelson-Oppen Method

Now we can explain step one of the Nelson-Oppen method:

1. Conversion to Separate Form

Given a conjunction of literals, $\phi$, we desire to convert it into a separate form: a $T$-equisatisfiable conjunction of literals $\phi_1 \land \phi_2 \land \ldots \land \phi_n$, where each $\phi_i$ is $i$-pure.

The following algorithm accomplishes this.

1. Let $\psi$ be some literal in $\phi$.

2. If $\psi$ is $i$-pure, for some $i$, remove $\psi$ from $\phi$ and add $\psi$ to $\phi_i$; if $\phi$ is empty then stop; otherwise go to step 1.

3. Let $t$ be a non-variable term in $\psi$. Replace $t$ in $\phi$ with a new variable $z$, and add $z = t$ to $\phi$. Go to step 1.
The Nelson-Oppen Method

It is easy to see that $\phi$ is $T$-satisfiable iff $\phi_1 \land \ldots \land \phi_n$ is $T$-satisfiable.

Furthermore, because each $\phi_i$ is a $\Sigma_i$-formula, we can run $Sat_i$ on each $\phi_i$.

Clearly, if $Sat_i$ reports that any $\phi_i$ is unsatisfiable, then $\phi$ is unsatisfiable.

But the converse is not true in general.

**Example:** $f(0) \neq f(1 - 1)$

We need a way for the decision procedures to communicate with each other about shared variables.
The Nelson-Oppen Method

Suppose that $T_1$ and $T_2$ are theories with disjoint signatures $\Sigma_1$ and $\Sigma_2$ respectively. Let $T = Cn \bigcup T_i$ and $\Sigma = \bigcup \Sigma_i$. Given a $\Sigma$-formula $\phi$ and decision procedures $Sat_1$ and $Sat_2$ for $T_1$ and $T_2$ respectively, we wish to determine if $\phi$ is $T$-satisfiable. The non-deterministic Nelson-Oppen algorithm for this is as follows:

1. Convert $\phi$ to its separate form $\phi_1 \land \phi_2$.

2. Let $\Lambda$ be the set of variables shared between $\phi_1$ and $\phi_2$. Guess an equivalence relation $\sim$ on $\Lambda$.

3. Run $Sat_1$ on $\phi_1 \cup Ar_{\sim}$.

4. Run $Sat_2$ on $\phi_2 \cup Ar_{\sim}$.

If there exists an equivalence relation $\sim$ such that both $Sat_1$ and $Sat_2$ succeed, then we claim that $\phi$ is $T$-satisfiable.

If no such equivalence relation exists, then we claim that $\phi$ is $T$-unsatisfiable.

The generalization to more than two theories is straightforward.
Example

Consider the combination of the theory $T_Z$ with the theory $T_E$ of equality.

Let $\phi = 1 \leq x \land x \leq 2 \land f(x) \neq f(1) \land f(x) \neq f(2)$.

Is this satisfiable?
Example

Consider the combination of the theory $T_Z$ with the theory $T_E$ of equality.

Let $\phi = 1 \leq x \land x \leq 2 \land f(x) \neq f(1) \land f(x) \neq f(2)$.

Is this satisfiable? No.
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Let $\phi = 1 \leq x \land x \leq 2 \land f(x) \neq f(1) \land f(x) \neq f(2)$.

Is this satisfiable? **No.**

To determine this using the above algorithm, we first convert $\phi$ to a separate form:

$\phi_Z = 1 \leq x \land x \leq 2 \land y = 1 \land z = 2$
$\phi_E = f(x) \neq f(y) \land f(x) \neq f(z)$

Now, the shared variables are $\{x, y, z\}$. There are 5 possible arrangements based on equivalence classes of $x, y,$ and $z$:

1. $\{x = y, x = z, y = z\}$
2. $\{x = y, x \neq z, y \neq z\}$
3. $\{x \neq y, x = z, y \neq z\}$
4. $\{x \neq y, x \neq z, y = z\}$
5. $\{x \neq y, x \neq z, y \neq z\}$
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1. $\{x = y, x = z, y = z\}$: inconsistent with $\phi_E$.
2. $\{x = y, x \neq z, y \neq z\}$
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4. $\{x \neq y, x \neq z, y = z\}$: inconsistent with $\phi_Z$.
5. $\{x \neq y, x \neq z, y \neq z\}$: inconsistent with $\phi_Z$. 
Example

Consider the combination of the theory $T_Z$ with the theory $T_E$ of equality.

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3. $\{x \neq y, x = z, y \neq z\}$: inconsistent with $\phi_E$.
4. $\{x \neq y, x \neq z, y = z\}$: inconsistent with $\phi_Z$.
5. $\{x \neq y, x \neq z, y \neq z\}$: inconsistent with $\phi_Z$.

This example is in combining.ml.
Correctness of Nelson-Oppen

We define an interpretation of a signature $\Sigma$ to be a model of $\Sigma$ together with a variable assignment.

Two interpretations $A$ and $B$ are isomorphic if there exists an isomorphism $h$ of the model of $A$ into the model of $B$ and $h(x^A) = x^B$ for each variable $x$ (where $x^A$ signifies the value assigned to $x$ by the variable assignment of $A$).

We furthermore define $A^{\Sigma,V}$ to be the restriction of $A$ to the symbols in $\Sigma$ and the variables in $V$.

**Theorem**

Let $\Sigma_1$ and $\Sigma_2$ be signatures, and for $i = 1, 2$, let $\phi_i$ be a set of $\Sigma_i$-formulas, and $V_i$ the set of variables appearing in $\phi_i$. Then $\phi_1 \cup \phi_2$ is satisfiable iff there exists a $\Sigma_1$-interpretation $A$ satisfying $\phi_1$ and a $\Sigma_2$-interpretation $B$ satisfying $\phi_2$ such that:

$$A^{\Sigma_1 \cap \Sigma_2, V_1 \cap V_2} \text{ is isomorphic to } B^{\Sigma_1 \cap \Sigma_2, V_1 \cap V_2}.$$
Correctness of Nelson-Oppen

Proof

Let \( \Sigma = \Sigma_1 \cap \Sigma_2 \) and \( V = V_1 \cap V_2 \).

Suppose \( \phi_1 \cup \phi_2 \) is satisfiable. Let \( M \) be an interpretation satisfying \( \phi_1 \cup \phi_2 \). If we let \( A = M^{\Sigma_1,V_1} \) and \( B = M^{\Sigma_2,V_2} \), then clearly

- \( A \models \phi_1 \)
- \( B \models \phi_2 \)
- \( A^{\Sigma,V} \) is isomorphic to \( B^{\Sigma,V} \)

On the other hand, suppose that we have \( A \) and \( B \) satisfying the three conditions listed above. Let \( h \) be an isomorphism from \( A^{\Sigma,V} \) to \( B^{\Sigma,V} \).

We define an interpretation \( M \) as follows:

- \( \text{dom}(M) = \text{dom}(A) \)
- For each variable or constant \( u \), \( u^M = \begin{cases} u^A & \text{if } u \in (\Sigma^C_1 \cup V_1) \\ h^{-1}(u^B) & \text{otherwise} \end{cases} \)
Correctness of Nelson-Oppen

- For function symbols of arity $n$,
  \[
  f^M(a_1, \ldots, a_n) = \begin{cases} 
  f^A(a_1, \ldots, a_n) & \text{if } f \in \Sigma^F_1 \\
  h^{-1}(f^B(h(a_1), \ldots, h(a_n))) & \text{otherwise}
  \end{cases}
  \]

- For predicate symbols of arity $n$,
  \[
  (a_1, \ldots, a_n) \in P^M \iff (a_1, \ldots, a_n) \in P^A \quad \text{if } P \in \Sigma^P_1 \\
  (a_1, \ldots, a_n) \in P^M \iff (h(a_1), \ldots, h(a_n)) \in P^B \quad \text{otherwise}
  \]

By construction, $M^{\Sigma_1, V_1}$ is isomorphic to $A$. In addition, it is easy to verify that $h$ is an isomorphism of $M^{\Sigma_2, V_2}$ to $B$.

It follows by the homomorphism theorem that $M$ satisfies $\phi_1 \cup \phi_2$.  
\[\square\]
Correctness of Nelson-Oppen

Theorem

Let $\Sigma_1$ and $\Sigma_2$ be signatures, with $\Sigma_1 \cap \Sigma_2 = \emptyset$, and for $i = 1, 2$, let $\phi_i$ be a set of $\Sigma_i$-formulas, and $V_i$ the set of variables appearing in $\phi_i$. As before, let $V = V_1 \cap V_2$. Then $\phi_1 \cup \phi_2$ is satisfiable iff there exists an interpretation $A$ satisfying $\phi_1$ and an interpretation $B$ satisfying $\phi_2$ such that:

1. $|A| = |B|$, and
2. $x^A = y^A$ iff $x^B = y^B$ for every pair of variables $x, y \in V$.

Proof

Clearly, if $\phi_1 \cup \phi_2$ is satisfiable in some interpretation $M$, then the only if direction holds by letting $A = M$ and $B = M$.

Consider the converse. Let $h : V^A \to V^B$ be defined as $h(x^A) = x^B$. This definition is well-formed by property 2 above.

In fact, $h$ is bijective. To show that $h$ is injective, let $h(a_1) = h(a_2)$. Then there exist variables $x, y \in V$ such that $a_1 = x^A$, $a_2 = y^A$, and $x^B = y^B$. By property 2, $x^A = y^A$, and therefore $a_1 = a_2$. 
Correctness of Nelson-Oppen

To show that $h$ is surjective, let $b \in V^B$. Then there exists a variable $x \in V^B$ such that $x^B = b$. But then $h(x^A) = b$.

Since $h$ is bijective, it follows that $|V^A| = |V^B|$, and since $|A| = |B|$, we also have that $|A - V^A| = |B - V^B|$. We can therefore extend $h$ to a bijective function $h'$ from $A$ to $B$.

By construction, $h'$ is an isomorphism of $A^V$ to $B^V$. Thus, by the previous theorem, we can obtain an interpretation satisfying $\phi_1 \cup \phi_2$. 

$\square$
Correctness of Nelson-Oppen

We can finally prove the correctness of the nondeterministic Nelson-Oppen method.

**Theorem**

Let $T_i$ be a stably-infinite $\Sigma_i$-theory, for $i = 1, 2$, and suppose that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Also, let $\phi_i$ be a set of $\Sigma_i$ literals, $i = 1, 2$, and let $\Lambda$ be the set of variables appearing in both $\phi_1$ and $\phi_2$. Then $\phi_1 \cup \phi_2$ is $T_1 \cup T_2$-satisfiable iff there exists an equivalence relation $\sim$ on $\Lambda$ such that $\phi_i \cup Ar_{\sim}$ is $T_i$-satisfiable, $i = 1, 2$.

**Proof**

Suppose $M$ is an interpretation satisfying $\phi_1 \cup \phi_2$. We define an equivalence relation $x \sim y$ iff $x, y \in \Lambda$ and $x^M = y^M$. By construction, $M$ is a $T_i$-interpretation satisfying $\phi_i \cup Ar_{\sim}, i = 1, 2$. 
Correctness of Nelson-Oppen

Suppose on the other hand that there exists an equivalence relation $\sim$ of $\Lambda$ such that $\phi_i \cup Ar \sim$ is $T_i$-satisfiable, $i = 1, 2$. Since $T_1$ is stably-infinite, there is an infinite interpretation $A$ satisfying $\phi_1 \cup Ar \sim$. Similarly, there is an infinite interpretation $B$ satisfying $\phi_2 \cup Ar \sim$.

But by LST, we can take the least upper bound of $|A|$ and $|B|$ and obtain interpretations of that cardinality.

Then we have $|A| = |B|$ and $x^A = y^A$ iff $x^B = y^B$ for every variable $x, y \in \Lambda$. We can thus apply the previous theorem and obtain the existence of a $(\Sigma_1 \cup \Sigma_2)$-interpretation satisfying $\phi_1 \cup \phi_2$. 

$\square$