Review

Last time

- Compactness
- Enumerability Theorem
- Definability of Models
- Finite Models
- Size of Models
Outline

- Theories
- Congruence Closure
- Interpretations Between Theories

Sources:

Sections 2.6 through 2.7 of Enderton.


Theories

Last time, we defined a *theory* as a set of first-order sentences.

For this lecture we will refine our definition to be a set of first-order sentences *closed under logical implication*.

Thus, $T$ is a theory iff $T$ is a set of sentences and if $T \models \sigma$, then $\sigma \in T$ for every sentence $\sigma$.

**Examples**

- For a given signature, the smallest possible theory consists of exactly the valid sentences over that signature.

- The largest theory for a given signature is the set of all sentences. It is the only unsatisfiable theory. Why?
Theories

For a class $\mathcal{K}$ of models over a given signature $\Sigma$, define the theory of $\mathcal{K}$ as

$$Th\mathcal{K} = \{\sigma \mid \sigma \text{ is a } \Sigma\text{-sentence which is true in every model in } \mathcal{K}\}.$$

**Theorem**

$Th\mathcal{K}$ is indeed a theory.

**Proof**

Suppose $Th\mathcal{K} \models \sigma$. We know that $\models_M Th\mathcal{K}$ for each $M$ in $\mathcal{K}$. It follows that $\models_M \sigma$ for each $M$ in $\mathcal{K}$, and thus $\sigma \in Th\mathcal{K}$.

Suppose $\Gamma$ is a set of sentences.

Define the set $Cn \Gamma$ of consequences of $\Gamma$ to be $\{\sigma \mid \Gamma \models \sigma\}$.

Then $Cn \Gamma = Th\ Mod \Gamma$. 
Theories

A theory $T$ is **complete** iff for every sentence $\sigma$, either $\sigma \in T$ or $(\neg \sigma) \in T$.

Note that if $M$ is a model, then $\text{Th } \{M\}$ is complete. In fact, for a class $\mathcal{K}$ of models, $\text{Th } \mathcal{K}$ is complete iff any two members of $\mathcal{K}$ are elementarily equivalent.

A theory $T$ is **axiomatizable** iff there is a decidable set $\Gamma$ of sentences such that $T = \text{Cn } \Gamma$.

A theory $T$ is **finitely axiomatizable** iff $T = \text{Cn } \Gamma$ for some finite set $\Gamma$ of sentences.

**Theorem**

If $\text{Cn } \Gamma$ is finitely axiomatizable, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\text{Cn } \Gamma_0 = \text{Cn } \Gamma$.

**Proof**

If $\text{Cn } \Gamma$ is finitely axiomatizable, then for some sentence $\tau$, $\text{Cn } \Gamma = \text{Cn } \tau$.

Clearly, $\Gamma \models \tau$. By compactness, we have that there exists $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \tau$. Thus, $\text{Cn } \tau \subseteq \text{Cn } \Gamma_0 \subseteq \text{Cn } \Gamma$, and since $\text{Cn } \Gamma = \text{Cn } \tau$, it follows that $\text{Cn } \Gamma_0 = \text{Cn } \Gamma$.
Theories

Using the above terminology, we can restate our earlier results as follows:

- An axiomatizable theory (in a reasonable language) is effectively enumerable.
- A complete axiomatizable theory (in a reasonable language) is decidable.

Our results about theories can be summarized in the following diagram.
Los-Vaught Test

For a theory $T$ and a cardinal $\lambda$, say that $T$ is $\lambda$-categorical iff all models of $T$ having cardinality $\lambda$ are isomorphic.

**Theorem**

Let $T$ be a theory in a countable language such that

- $T$ is $\lambda$-categorical for some infinite cardinal $\lambda$.
- All models of $T$ are infinite.

Then $T$ is complete.

**Proof**

It suffices to show that for any two models $M$ and $M'$ of $T$, $M \equiv M'$. Since $M$ and $M'$ are infinite, there exist (by LST) elementarily equivalent models of cardinality $\lambda$. But these models must be isomorphic, and by the homomorphism theorem, isomorphic models are elementarily equivalent.
Prenex Normal Form

A prenex formula is one of the form $Q_1 x_1 \cdots Q_n x_n \alpha$, where each $Q_i$ is a quantifier and $\alpha$ is quantifier-free.

Theorem

For any formula we can find a logically equivalent prenex formula.

Proof

We use the following quantifier identities which are easy to prove.

- $\neg \forall x \alpha \leftrightarrow \exists x \neg \alpha$
- $\neg \exists x \alpha \leftrightarrow \forall x \neg \alpha$
- $(\alpha \rightarrow \forall x \beta) \leftrightarrow \forall x (\alpha \rightarrow \beta)$, for $x$ not free in $\alpha$.
- $(\alpha \rightarrow \exists x \beta) \leftrightarrow \exists x (\alpha \rightarrow \beta)$, for $x$ not free in $\alpha$.
- $(\forall x \alpha \rightarrow \beta) \leftrightarrow \exists x (\alpha \rightarrow \beta)$, for $x$ not free in $\beta$.
- $(\exists x \alpha \rightarrow \beta) \leftrightarrow \forall x (\alpha \rightarrow \beta)$, for $x$ not free in $\beta$.

By using these identities together with renaming of bound variables, it is easy to manipulate any formula to a logically equivalent prenex formula.
Validity and Satisfiability Modulo Theories

Given a $\Sigma$-theory $T$, a $\Sigma$-formula $\phi$ is

1. $T$-valid if $\models_M \phi[s]$ for all models $M$ of $T$ and all variable assignments $s$.

2. $T$-satisfiable if there exists some model $M$ of $T$ and variable assignment $s$ such that $\models_M \phi[s]$.

3. $T$-unsatisfiable if $\not\models_M \phi[s]$ for all models $M$ of $T$ and all variable assignments $s$.

The validity problem for $T$ is the problem of deciding, for each $\Sigma$-formula $\phi$, whether $\phi$ is $T$-valid.

The satisfiability problem for $T$ is the problem of deciding, for each $\Sigma$-formula $\phi$, whether $\phi$ is $T$-satisfiable.

Similarly, one can define the quantifier-free validity problem and the quantifier-free satisfiability problem for a $\Sigma$-theory $T$ by restricting the formula $\phi$ to be quantifier-free.
Validity and Satisfiability Modulo Theories

A decision problem is *decidable* if there exists an effective procedure which always terminates with an answer for any given instance of the problem.

For example, the validity problem for a \( \Sigma \)-theory \( T \) is decidable if there exists an effective procedure for determining whether \( T \models \phi \) for every \( \Sigma \)-formula \( \phi \).

Note that validity problems can always be reduced to satisfiability problems:

\[ \phi \text{ is } T\text{-valid iff } \neg \phi \text{ is } T\text{-unsatisfiable.} \]

We will consider a few examples of theories which are of particular interest in verification applications.
The Theory $T_\mathcal{E}$ of Equality

The theory $T_\mathcal{E}$ of equality is the theory $Cn \emptyset$.

Note that the exact set of sentences in $T_\mathcal{E}$ depends on the signature in question.

A well-known result in first-order logic is that the general validity problem for $T_\mathcal{E}$ is undecidable.

However, the quantifier-free validity problem for $T_\mathcal{E}$ is decidable.
The Theory $T_Z$ of Integers

Let $\Sigma_Z$ be the signature $(0, 1, +, -, \leq)$. Let $A_Z$ be the standard model of the integers with domain $\mathbb{Z}$. Then $T_Z$ is defined to be $ThA_Z$.

As showed by Presburger in 1929, the validity problem for $T_Z$ is decidable, but its complexity is triply-exponential.

The quantifier-free validity problem for $T_Z$ is “only” NP-complete.

Let $\Sigma_Z^\times$ be the same as $\Sigma_Z$ with the addition of the symbol $\times$ for multiplication, and define $A_Z^\times$ and $T_Z^\times$ in the obvious way.

The validity problem for $T_Z^\times$ is undecidable and is a consequence of Gödel’s incompleteness theorem.

In fact, even the quantifier-free validity problem for $T_Z^\times$ is undecidable.
The Theory $T_R$ of Reals

Let $\Sigma_R$ be the signature $(0, 1, +, -, \leq)$.

Let $\mathcal{A}_R$ be the standard model of the reals with domain $\mathcal{R}$.

Then $T_R$ is defined to be $\text{Th} \mathcal{A}_R$.

The validity problem for $T_R$ is decidable.

The quantifier-free satisfiability problem for conjunctions of literals (atomic formulas or their negations) in $T_R$ is solvable in polynomial time, though exponential methods (like simplex or Fourier-Motzkin) tend to perform best in practice.

Let $\Sigma_R^\times$ be the same as $\Sigma_R$ with the addition of the symbol $\times$ for multiplication, and define $\mathcal{A}_R^\times$ and $T_R^\times$ in the obvious way.

In contrast to the theory of integers, the validity problem for $T_R^\times$ is decidable.
The Theory $T_L$ of Lists

Let $\Sigma_L$ be the signature ($\text{cons}, \text{car}, \text{cdr}$).

Let $\Lambda_L$ be the following axioms:

1. A construction axiom:
   \[ \forall x \ (\text{cons}(\text{car}(x), \text{cdr}(x)) = x). \]

2. Two selection axioms:
   
   \[ \forall x \forall y (\text{car}(\text{cons}(x, y)) = x) \]
   
   \[ \forall x \forall y (\text{cdr}(\text{cons}(x, y)) = y) \]

3. An infinite number of acyclicity axioms:

   \[ \text{car}(x) \neq x \]
   
   \[ \text{cdr}(x) \neq x \]
   
   \[ \text{car} \left( \text{car}(x) \right) \neq x \]
   
   \[ \text{car} \left( \text{cdr}(x) \right) \neq x \]

   \[ \ldots \]

Then $T_L = Cn \ \Lambda_L$. 
The Theory $T_L$ of Lists

The validity problem for $T_L$ is decidable but not elementary recursive.

This means there is no decision procedure for the $T_L$-validity of $\Sigma_L$-formulas that always stops in time $2^2 \cdots 2^n$ for any fixed number of 2’s.

More reasonable complexity results hold for the quantifier-free case.

The problem of deciding the $T_L$-satisfiability of conjunctions of $\Sigma_L$-literals is solvable in linear time.
The Theory $T_A$ of Arrays

Let $\Sigma_A$ be the signature $(\text{read}, \text{write})$.

Let $\Lambda_A$ be the following axioms:

\begin{align*}
\forall a \forall i \forall v \left( \text{read}(\text{write}(a, i, v), i) = v \right) \\
\forall a \forall i \forall j \forall v \left( i \neq j \rightarrow \text{read}(\text{write}(a, i, v), j) = \text{read}(a, j) \right) \\
\forall a \forall b \left( \left( \forall i \left( \text{read}(a, i) = \text{read}(b, i) \right) \right) \rightarrow a = b \right)
\end{align*}

Then $T_A = Cn \Lambda_A$.

The validity problem for $T_A$ is undecidable, but the quantifier-free validity problem for $T_A$ is decidable (its complexity is NP).
Congruence Closure

Let $G = (V, E)$ be a directed graph such that for each vertex $v$ in $G$, the successors of $v$ are ordered.

Let $C$ be any equivalence relation on $V$.

The congruence closure $C^*$ of $C$ is the finest equivalence relation on $V$ that contains $C$ and satisfies the following property for all vertices $v$ and $w$.

Let $v$ and $w$ have successors $v_1, \ldots, v_k$ and $w_1, \ldots, w_l$ respectively. If $k = l$ and $(v_i, w_i) \in C^*$ for $1 \leq i \leq k$, then $(v, w) \in C^*$.

In other words, if the corresponding successors of $v$ and $w$ are equivalent under $C^*$, then $v$ and $w$ are themselves equivalent under $C^*$.

Often, the vertices are labeled by some labeling function $\lambda$. In this case, the property becomes:

If $\lambda(v) = \lambda(w)$ and if $k = l$ and $(v_i, w_i) \in C^*$ for $1 \leq i \leq k$, then $(v, w) \in C^*$.
A Simple Algorithm

Let $C_0 = C$ and $i = 0$.

1. Number the equivalence classes in $C_i$ consecutively from 1.

2. Let $\alpha$ assign to each vertex $v$ the number $\alpha(v)$ of the equivalence class containing $v$.

3. For each vertex $v$ construct a signature $s(v) = \lambda(v)(\alpha(v_1), \ldots, \alpha(v_k))$, where $v_1, \ldots, v_k$ are the successors of $v$.

4. Group the vertices into classes of vertices having equal signatures.

5. Let $C_{i+1}$ be the finest equivalence relation on $V$ such that two vertices equivalent under $C_i$ or having the same signature are equivalent under $C_{i+1}$.

6. If $C_{i+1} = C_i$, let $C^* = C_i$; otherwise increment $i$ and repeat.
Congruence Closure and $T_\mathcal{E}$

Recall that $T_\mathcal{E}$ is the empty theory with equality over some signature $\Sigma$ containing only function symbols.

If $\Gamma$ is a set of ground $\Sigma$-equalities and $\Delta$ is a set of ground $\Sigma$-disequalities, then the satisfiability of $\Gamma \cup \Delta$ can be determined as follows.

- Let $G$ be a graph which corresponds to the abstract syntax trees of terms in $\Gamma \cup \Delta$, and let $v_t$ denote the vertex of $G$ associated with the term $t$.
- Let $C$ be the equivalence relation on the vertices of $G$ induced by $\Gamma$.
- $\Gamma \cup \Delta$ is satisfiable iff for each $s \neq t \in \Delta$, $(v_s, v_t) \notin C^*$.
An Algorithm for $T_\mathcal{E}$

union and find are abstract operations for manipulating equivalence classes.

union$(x, y)$ merges the equivalence classes of $x$ and $y$.

find$(x)$ returns a unique representative of the equivalence class of $x$.

$CC(\Gamma, \Delta)$

Construct $G(V, E)$ from terms in $\Gamma$ and $\Delta$.

while $\Gamma \neq \emptyset$

    Remove some equality $a = b$ from $\Gamma$;
    
    Merge$(a, b)$;
    
    if find$(a) = \text{find}(b)$ for some $a \neq b \in \Delta$ then

        return false;
    
    return true;
An Algorithm for $T_{\mathcal{C}}$

$\text{Merge}(a, b)$

if $\text{find}(a) = \text{find}(b)$ then return;

Let $A$ be the set of all predecessors of all vertices equivalent to $a$;
Let $B$ be the set of all predecessors of all vertices equivalent to $b$;
$\text{union}(a, b)$;
foreach $x \in A$ and $y \in B$
  if $\text{signature}(x) = \text{signature}(y)$ then $\text{Merge}(x, y)$;
Congruence Closure

DST Algorithm

The Downey-Sethi-Tarjan Congruence Closure algorithm is more efficient. It makes use of some additional data structures and methods.

Additional Helper Methods

- \( \text{union}(a, b) \) in this algorithm, the \textit{first} argument always becomes the new equivalence class representative.
- \( \text{list}(e) \) returns the list of vertices with at least one successor in equivalence class \( e \).
- \( \text{enter}(v) \) stores \((v, \text{signature}(v))\) in a signature table.
- \( \text{delete}(v) \) removes \((v, \text{signature}(v))\) from the signature table if it is there. Note that this operation does \textit{not} remove any other entry, even if it has the same signature as \( v \).
- \( \text{query}(v) \) if there is an entry \((w, \text{signature}(w))\) in the signature table, and \( \text{signature}(w) = \text{signature}(v) \), then return \( w \); otherwise, return \( \bot \).
DST Algorithm

\( cc(\Gamma, \Delta) \)

Construct \( G(V, E) \) from terms in \( \Gamma \) and \( \Delta \).

\( \text{Merge}(\Gamma); \)

if \( \text{find}(a) = \text{find}(b) \) for some \( a \neq b \in \Delta \) then

\[ \text{return } \text{false}; \]

return \( \text{true} \);
**DST Algorithm**

\[ \text{Merge}(\text{combine}) \]

\[
\begin{align*}
\text{pending} & := \text{set of all vertices}; \\
\text{while } \text{pending} \neq \emptyset \\
\text{foreach } v \in \text{pending} \\
\quad \text{if } \text{query}(v) = \perp \text{ then } \text{enter}(v); \\
\quad \text{else } \text{add } (v, \text{query}(v)) \text{ to } \text{combine}; \\
\text{pending} & := \emptyset; \\
\text{foreach } (a, b) \in \text{combine} \\
\quad \text{if } \text{find}(a) \neq \text{find}(b) \text{ then} \\
\quad \quad \text{if } |\text{list(find}(a))| < |\text{list(find}(b))| \text{ then swap } a \text{ and } b; \\
\quad \text{foreach } u \in \text{list(find}(b)) \\
\quad \quad \text{delete}(u); \text{ add } u \text{ to } \text{pending}; \\
\quad \quad \text{union}(\text{find}(a), \text{find}(b)); \\
\text{combine} & := \emptyset;
\end{align*}
\]
Interpretations Between Theories

Given two theories, $T_0$ and $T_1$, sometimes it is possible to show that one of the theories is at least as powerful as the other.

A simple case is when $T_0$ and $T_1$ are in the same language and one is a subset of the other.

We will consider more general cases in which the languages of the two theories differ.
Defining Functions

A common practice in mathematical reasoning is to introduce a new piece of notation, defining it in terms of a formula not containing the new notation.

Formally, suppose $\Sigma$ is a signature and $f$ is a function symbol not in $\Sigma$. Let $\Sigma^+ = \Sigma \cup \{f\}$. A function definition is a formula of the form:

$$\forall v_1 \forall v_2 (fv_1 = v_2 \leftrightarrow \phi)$$

where $\phi$ is a $\Sigma$-formula in which only $v_1$ and $v_2$ may occur free. Let the above sentence be designated $\delta$.

Theorem

The following are equivalent:

1. The definition is non-creative, i.e. for any $\Sigma$-sentence $\sigma$ if $T \cup \{\delta\} \models \sigma$ in $\Sigma^+$, then $T \models \sigma$ in $\Sigma$.

2. $f$ is well-defined, i.e. the following sentence (which we designate $\epsilon$) is in the theory $T$:

$$\forall v_1 \exists! v_2 \phi.$$ 

Note that $\exists!$ means “there exists uniquely”. Technically it is an abbreviation for a more complicated formula which expresses both existence and uniqueness.
**Defining Functions**

**Proof**

To show that (1) implies (2), note that $\delta \models \varepsilon$. Thus, if we take $\sigma \equiv \varepsilon$ in (1), it follows that $T \models \varepsilon$.

To show that (2) implies (1), suppose that $T \models \varepsilon$. Let $M$ be a $\Sigma$-model of $T$. For $d \in \text{dom}(M)$, let $F(d)$ be the unique $e \in \text{dom}(M)$ such that $M \models \phi[[d, e]]$. The existence of such an $e$ for each $d$ is ensured because $T \models \varepsilon$. Now let $(M, F)$ be the $\Sigma^+$ model which agrees with $M$ on all parameters except $F$ and which assigns $F$ to the symbol $f$. It is easy to see that $(M, F)$ is a model of $\delta$.

Thus, if $T \cup \{\delta\} \models \sigma$, then $(M, F) \models \sigma$. But since $\sigma$ is a $\Sigma$-sentence, it follows that $M \models \sigma$. Since $M$ was chosen arbitrarily, it follows that $T \models \sigma$. 

□
Interpretations

There are more general ways in which one theory can be as strong as another theory in another language.

Example

Consider the theory of $\langle \mathcal{N}, 0, S \rangle$ (natural numbers with 0 and successor) and on the other hand the theory of $\langle \mathbb{Z}, +, \times \rangle$.

The second theory is at least as strong as the first. To show this, we make the following observations:

- An integer is nonnegative iff it is the sum of four squares.
- The set $\{0\}$ is defined by $\nu_1 + \nu_1 = \nu_1$.
- The successor relation is defined by $\forall \, z \, (z \times z = z \land z + z \neq z \rightarrow \nu_1 + z = \nu_2)$.

Thus, for example, the sentence $\forall \, x \, Sx \neq 0$ can be translated as

$$\forall \, x \, [\exists \, y_1 \exists \, y_2 \exists \, y_3 \exists \, y_4 \, x = y_1 \times y_1 + y_2 \times y_2 + y_3 \times y_3 + y_4 \times y_4 \rightarrow \neg \forall \, u \, (u + u = u \rightarrow \forall \, v \, (\forall \, z \, (z \times z = z \land z + z \neq z \rightarrow x + z = v) \rightarrow v = u))].$$
Interpretations

Suppose $\Sigma_0$ and $\Sigma_1$ are signatures and $T_1$ is a $\Sigma_1$-theory.

An interpretation $\pi$ of $\Sigma_0$ into $T_1$ consists of the following three items.

1. a $\Sigma_1$-formula $\pi_\forall$ in which at most $v_1$ occurs free, such that
   (i) $T_1 \models \exists v_1 \pi_\forall$.

2. a $\Sigma_1$-formula $\pi_P$ for each $n$-ary predicate symbol $P \in \Sigma_0$ in which at most
   the variables $v_1, \ldots, v_n$ occur free.

3. a $\Sigma_1$-formula $\pi_f$ for each $n$-ary function symbol $f \in \Sigma_0$ in which at most
   $v_1, \ldots, v_n, v_{n+1}$ occur free such that
   (ii) $T_1 \models \forall v_1, \ldots, v_n (\pi_\forall(v_1) \to \cdots \to \pi_\forall(v_n)$
   $\to \exists x (\pi_\forall(x) \land \forall v_{n+1} (\pi_f(v_1, \ldots, v_{n+1}) \leftrightarrow v_{n+1} = x)))$.

For our previous example, we have

1. $\pi_\forall(x) = \exists y_1 \exists y_2 \exists y_3 \exists y_4 x = y_1 \times y_1 + y_2 \times y_2 + y_3 \times y_3 + y_4 \times y_4$

2. $\pi_0(x) = x + x = x$

3. $\pi_S(x, y) = \forall y (z \times z \land z + z \neq z \to x + z = y)$
Interpretations

Assume that \( \pi \) is an interpretation and let \( M \) be a model of \( T_1 \).

There is a natural way to extract from \( M \) a model \( \pi M \) for \( \Sigma_0 \). Let

- \( \text{dom}(\pi M) = \) the set defined in \( M \) by \( \pi \forall \),
- \( P^{\pi M} = \) the relation defined in \( M \) by \( \pi P \), restricted to \( \text{dom}(\pi M) \).
- \( f^{\pi M}(a_1, \ldots, a_n) = \) the unique \( b \) such that \( M \models \pi f[[a_1, \ldots, a_n, b]] \), where \( a_1, \ldots, a_n \) are in \( \text{dom}(\pi M) \).

By condition (i), \( \text{dom}(\pi M) \neq \emptyset \). By condition (ii), the definition of \( f^{\pi M} \) makes sense.

Define the set \( \pi^{-1}[T_1] \) of \( \Sigma_0 \)-sentences as

\[
\text{Th}\{ \pi M \mid M \in \text{Mod}T_1 \}.
\]

In other words, \( \pi^{-1}[T_1] \) is the set of all \( \Sigma_0 \)-sentences true in every model obtainable from a model of \( T_1 \) as shown above.
Interpretations

Given a $\Sigma_0$ formula $\phi$ and an interpretation $\pi$ of $\Sigma_0$ into $T_1$, we can find a formula $\phi^\pi$ which in some sense corresponds exactly to $\phi$.

For an atomic formula $\alpha$, we scan the formula from right to left. When a function symbol $f$ is found, applied to arguments $x_1, \ldots, x_n$, it is replaced by a new variable $y$ and the atomic formula is prefixed by $\forall y \left( \pi f(x_1, \ldots, x_n, y) \rightarrow \right)$. We continue until there are no more function symbols. Finally, we replace the predicate symbol $P$ (if it is not equality) by $\pi_P$.

For example,
\[
(P f g x)^\pi = \forall y \left( \pi_g(x, y) \rightarrow (P f y)^\pi \right) \\
= \forall y \left( \pi_g(x, y) \rightarrow \forall z(\pi_f(y, z) \rightarrow (P z)^\pi) \right) \\
= \forall y \left( \pi_g(x, y) \rightarrow \forall z(\pi_f(y, z) \rightarrow \pi_P(z)) \right).
\]

The interpretation of non-atomic formulas are defined in the obvious way:
\[
(\neg \phi)^\pi = (\neg \phi^\pi), \quad (\phi \rightarrow \psi)^\pi = (\phi^\pi \rightarrow \psi^\pi), \quad \text{and} \\
(\forall x \phi)^\pi = \forall x \left( \pi \forall(x) \rightarrow \phi^\pi \right).
\]
Interpretations

Lemma

Let \( \pi \) be an interpretation of \( \Sigma_0 \) into \( T_1 \) and \( M \) a model of \( T_1 \). For any \( \Sigma_0 \)-formula \( \phi \) and any map \( s \) of the variables into \( \text{dom}(\pi M) \),

\[
\models_{\pi M} \phi[s] \iff \models_M \phi^\pi[s].
\]

The proof is by induction on \( \phi \) and is omitted.

Corollary

For a \( \Sigma_0 \)-sentence \( \sigma \), \( \sigma \in \pi^{-1}[T_1] \) iff \( \sigma^\pi \in T_1 \).

An interpretation \( \pi \) of a theory \( T_0 \) into a theory \( T_1 \) is an interpretation \( \pi \) of the signature of \( T_0 \) into \( T_1 \) such that \( T_0 \subseteq \pi^{-1}[T_1] \).

If \( T_0 = \pi^{-1}[T_1] \), then \( \pi \) is said to be a faithful interpretation of \( T_0 \) into \( T_1 \).

For a faithful interpretation, we have \( \sigma \in T_0 \) iff \( \sigma^\pi \in T_1 \).