Review

Last time

- Proofs
- A Deductive Calculus
- Deduction Rules
- Soundness

Outline

- Homomorphisms
- Completeness

Sources:

Sections 2.2 and 2.5 of Enderton.

Soundness and Completeness

An important question for any calculus is its relationship to the semantic notion of validity.

If only valid formulas are deducible, then the calculus is said to be sound.

If all valid formulas are deducible, then the calculus is said to be complete.

The existence of a sound and complete calculus for first-order logic is an important result which demonstrates that it is a reasonable model of mathematical thinking.
Soundness and Completeness

Soundness Theorem
If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.

Completeness Theorem (Gödel, 1930)
If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.
This is equivalent to the following statement: any consistent set of formulas is satisfiable.

Homomorphisms
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are models over the same signature $\Sigma$.

A homomorphism $h$ of $\mathcal{A}$ into $\mathcal{B}$ is a function $h : \text{dom}(\mathcal{A}) \to \text{dom}(\mathcal{B})$ such that
1. For each $n$-ary predicate symbol $P \in \Sigma$ and each $n$-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $\text{dom}(\mathcal{A})$,
   if $\langle a_1, \ldots, a_n \rangle \in P^\mathcal{A}$, then $\langle h(a_1), \ldots, h(a_n) \rangle \in P^\mathcal{B}$.
2. For each $n$-ary function symbol $f \in \Sigma$ and each $n$-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $\text{dom}(\mathcal{A})$,
   $h(f^\mathcal{A}(a_1, \ldots, a_n)) = f^\mathcal{B}(h(a_1), \ldots, h(a_n))$.
3. For each constant symbol $c \in \Sigma$, $h(c^\mathcal{A}) = c^\mathcal{B}$.

If (1) also holds in reverse, then $h$ is a strong homomorphism (this is what the book calls a homomorphism).

A strong homomorphism which is injective (one-to-one) is an embedding.

An embedding which is surjective (onto) is an isomorphism.

A homomorphism of $\mathcal{A}$ into $\mathcal{A}$ is called an endomorphism of $\mathcal{A}$.

An isomorphism of $\mathcal{A}$ to $\mathcal{A}$ is called an automorphism of $\mathcal{A}$.

Example
Let $\mathcal{A} = (\mathcal{N}, +, \times)$, and let $\mathcal{B} = (\{0, 1\}, +_{\mathcal{B}}, \times)$.
Define $h : \mathcal{N} \to \{0, 1\}$ by: $h(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

Then $h$ is a homomorphism.

Proof
The proof is by cases. For example, suppose $a, b \in \mathcal{N}$ and both are odd.
Then $a + b$ is even, so $h(a + b) = 0$. Similarly, $h(a) = h(b) = 1$, so $h(a) +_{\mathcal{B}} h(b) = 0$.
Also, $a \times b$ is odd, so $h(a \times b) = 1$. Similarly, $h(a) \times h(b) = 1$.
The other cases are similar.

Substructures and Extensions
Let $\mathcal{P}$ be the set of positive integers.

Then there is an isomorphism $h$ from $(\mathcal{P}, <)$ to $(\mathcal{N}, <)$ defined by $h(n) = n - 1$.

Note also that the identity map is an embedding of $(\mathcal{P}, <)$ into $(\mathcal{N}, <)$.

Because of this, we say that $(\mathcal{P}, <)$ is a substructure of $(\mathcal{N}, <)$.

More generally, if $\mathcal{A}$ and $\mathcal{B}$ are models with $\text{dom}(\mathcal{A}) \subseteq \text{dom}(\mathcal{B})$ and the identity map $i : \text{dom}(\mathcal{A}) \to \text{dom}(\mathcal{B})$ is an embedding, then we say that $\mathcal{B}$ is an extension of $\mathcal{A}$ and that $\mathcal{A}$ is a substructure of $\mathcal{B}$.
Homomorphism Theorem

Let \( h \) be a strong homomorphism from \( \mathcal{A} \) to \( \mathcal{B} \), and let \( s \) map the set of variables into \( \text{dom}(\mathcal{A}) \).

1. For any term \( t, h(\overline{s}(t)) = \overline{h \circ s(t)} \), where \( \overline{s} \) is computed in \( \mathcal{A} \), and \( \overline{h \circ s(t)} \) is computed in \( \mathcal{B} \).

2. For any quantifier-free formula \( \alpha \) not containing the equality symbol,
   \[ \models_{\mathcal{A}} \alpha[s] \iff \models_{\mathcal{B}} \alpha[h \circ s]. \]

3. If \( h \) is an embedding, then the above holds even if \( \alpha \) contains the equality symbol.

4. If \( h \) is surjective (onto), then the above holds even if \( \alpha \) contains quantifiers.

Proof of Homomorphism Theorem

3. If \( h \) is an embedding, then for any quantifier-free formula \( \alpha \),
   \[ \models_{\mathcal{A}} \alpha[s] \iff \models_{\mathcal{B}} \alpha[h \circ s]. \]
   The argument for atomic formulas without equality is as above. For an equality, we have:
   \[ \models_{\mathcal{A}} u = t[s] \iff \overline{s}(u) = \overline{s}(t) \]
   \[ \iff \overline{h(\overline{s}(u))} = \overline{h(\overline{s}(t))} \quad h \text{ is 1-1.} \]
   \[ \iff \overline{h \circ s(u)} = \overline{h \circ s(t)} \quad \text{by (1).} \]
   \[ \iff \models_{\mathcal{B}} u = t[h \circ s]. \]
   As before, the more general case then follows by induction on \( \neg \) and \( \rightarrow \).

4. If \( h \) is surjective (onto), then the above statements hold even for formulas with quantifiers.

   In addition to the arguments given above, we must show an additional inductive case: if \( \alpha \) has the property that for every \( s, \models_{\mathcal{A}} \alpha[s] \iff \models_{\mathcal{B}} \alpha[h \circ s] \), then \( \forall x \alpha \) has this same property.

   Thus, we must show that for every \( s, \models_{\mathcal{A}} \forall x \alpha[s] \iff \models_{\mathcal{B}} \forall x \alpha[h \circ s] \).

Proof of Homomorphism Theorem

We must show that for every \( s, \models_{\mathcal{A}} \forall x \alpha[s] \iff \models_{\mathcal{B}} \forall x \alpha[h \circ s] \).

We start with the if direction. Suppose \( a \in \text{dom}(\mathcal{A}) \).

\[ \models_{\mathcal{B}} \forall x \alpha[h \circ s] \implies \models_{\mathcal{B}} \alpha[(h \circ s)(x|h(a))] \quad \text{semantics of } \forall \]
\[ \implies \models_{\mathcal{B}} \alpha[(h \circ s)(x|h(a))] \quad \text{by (1)} \]
\[ \implies \models_{\mathcal{A}} \alpha[s(x|a)] \quad \text{ind. hypothesis} \]

Since \( a \) was chosen arbitrarily, it follows that \( \models_{\mathcal{A}} \forall x \alpha[s] \).

For the other direction, suppose \( \not\models_{\mathcal{B}} \forall x \alpha[h \circ s] \).

\[ \not\models_{\mathcal{B}} \forall x \alpha[h \circ s] \implies \not\models_{\mathcal{B}} \alpha[(h \circ s)(x|h(b))] \quad \text{for some } b \in \text{dom}(\mathcal{B}) \]
\[ \implies \not\models_{\mathcal{B}} \alpha[(h \circ s)(x|h(a))] \quad h \text{ is onto} \]
\[ \implies \not\models_{\mathcal{B}} \alpha[(h \circ s)(x|h(a))] \quad \text{by (1)} \]
\[ \implies \models_{\mathcal{A}} \alpha[s(x|a)] \quad \text{ind. hypothesis} \]
\[ \implies \not\models_{\mathcal{A}} \forall x \alpha[s] \quad \text{semantics of } \forall \]

\( \square \)
**Automorphism Corollary**

As a corollary of the homomorphism theorem, we have the following:

**Theorem**

An automorphism must preserve definable relations: if \( h \) is an automorphism of \( \mathcal{A} \), and \( R \) is an \( n \)-ary relation on \( \text{dom}(\mathcal{A}) \) definable in \( \mathcal{A} \), then for any \( a_1, \ldots, a_n \in \text{dom}(\mathcal{A}) \),

\[
\langle a_1, \ldots, a_n \rangle \in R \iff \langle h(a_1), \ldots, h(a_n) \rangle \in R.
\]

**Proof**

Let \( \phi \) be the formula which defines \( R \) in \( \mathcal{A} \). By the homomorphism theorem,

\[
\models_{\mathcal{A}} \phi[s] \iff \models_{\mathcal{A}} \phi[h \circ s].
\]

It follows that

\[
\models_{\mathcal{A}} \phi[[a_1, \ldots, a_n]] \iff \models_{\mathcal{A}} \phi[[h(a_1), \ldots, h(a_n)]]
\]

and thus

\[
\langle a_1, \ldots, a_n \rangle \in R \iff \langle h(a_1), \ldots, h(a_n) \rangle \in R.
\]

\( \square \)

**Completeness**

**Completeness Theorem (Gödel, 1930)**

If \( \Gamma \models \phi \), then \( \Gamma \vdash \phi \).

This is equivalent to the following statement: any consistent set of formulas is satisfiable.

**Undefinable Relations**

We can sometimes use this corollary to show that a given relation is *not* definable.

Consider the model \( (\mathcal{R}, <) \) consisting of the set of real numbers with its usual ordering. We will show that the set \( \mathcal{N} \) of natural numbers is not definable in this model.

An automorphism of this model is any bijection from \( \mathcal{R} \) to \( \mathcal{R} \) which is strictly increasing:

\[
a < b \iff h(a) < h(b).
\]

One such function is \( h(a) = a^3 \).

Now, if \( \mathcal{N} \) were definable then by the above corollary we would have \( a \in \mathcal{N} \iff a^3 \in \mathcal{N} \) which is clearly untrue.