Review

- A Subtheory of Number Theory
- Representable Relations
- Church’s Thesis Revisited
- Representable Functions
- A Catalog of Representable Sets

Overview

Last time, we introduced the notion of representability and showed that many functions and relations are representable in $C_n\mathcal{A}_E$.

By encoding the syntax of first-order logic using natural numbers, we can encode facts about the terms and formulas of logic as relations in $\mathcal{N}$.

We can then use our results about representability to show that there are some surprising limits to what can be represented in $C_n\mathcal{A}_E$.

Outline

- Gödel Numbers
- Fixed-Point Lemma
- Tarski Undefinability Theorem
- Gödel Incompleteness Theorem

Source: Enderton, 3.4-3.5.
**Arithmetization of Syntax**

We begin by assigning a number to each symbol in our formal language.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Encoded symbol</th>
<th>Symbol</th>
<th>Encoded symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall )</td>
<td>0</td>
<td>(</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>)</td>
<td>3</td>
</tr>
<tr>
<td>S</td>
<td>4</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>&lt;</td>
<td>6</td>
<td>( \rightarrow )</td>
<td>7</td>
</tr>
<tr>
<td>+</td>
<td>8</td>
<td>=</td>
<td>9</td>
</tr>
<tr>
<td>\times</td>
<td>10</td>
<td>( v_1 )</td>
<td>11</td>
</tr>
<tr>
<td>( E )</td>
<td>12</td>
<td>( v_2 )</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( v_k )</td>
<td>9+2k</td>
</tr>
</tbody>
</table>

Note that this encoding could be modified to accommodate any countable signature (the symbols in the signature are assigned to the even numbers).

Let \( h \) be the function which maps each symbol to its encoding.

**Gödel Numbers**

For an expression \( e = s_0 \cdots s_n \) of the language, we define its Gödel number, \(#(e)#\) by

\[
#(s_0, \cdots, s_n) = \langle h(s_0), \ldots, h(s_n) \rangle.
\]

**Example**

\[
#(\exists v_3 \forall v_5 \ (\neg v_5 \right v_3 0))
\]

\[
= \langle 1, 5, 0, 15, 1, 5, 9, 15, 2, 3, 3 \rangle
\]

\[
= 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^{10} \cdot 11^2 \cdot 13^0 \cdot 17^{10} \cdot 19^1 \cdot 23^3 \cdot 29^4 \cdot 31^4.
\]

A set of expressions corresponds to a set of Gödel numbers:

\[
\#(\Phi) = \{ #(\epsilon) | \epsilon \in \Phi \}.
\]

**Gödel Numbers**

2. The set of Gödel numbers of terms is representable.

**Proof idea**

Let \( f \) be the corresponding characteristic function (i.e. the function whose value is 1 if its input is the Gödel number of a term and 0 otherwise).

Then,

\[
f(a) = \begin{cases} 
1 & \text{if } a \text{ is the Gödel number of a variable,} \\
1 & \text{if } \exists i < a \cdot \#(h(a)), \exists k < a \\
& \text{[i is a sequence number and} \\
& \text{\( k \) is the value of } h \text{ at some } \#(h(i))\text{-place function symbol and} \\
& \text{\( a = \langle k \rangle \ast * j < \#(h(i)) (i) \rangle \)} \\
0 & \text{otherwise.}
\end{cases}
\]
Gödel Numbers

11. The set of Gödel numbers of tautologies is representable.

17. The set of Gödel numbers of logical axioms is representable.

Let \( G(\langle \alpha_0, \ldots, \alpha_n \rangle) = \langle \#\alpha_0, \ldots, \#\alpha_n \rangle \).

18. For a finite set \( A \) of formulas, 

\[ \{ G(D) | D \text{ is a deduction from } A \} \]

is representable.

19. Any recursive relation is representable in \( C_n A_E \).

21. If \( \#A \) is recursive and \( Cn A \) is a complete theory, then \( \#Cn A \) is recursive.

Fixed-Point Lemma

Suppose \( \beta \) is a formula which defines some subset \( A \) of \( N \).

How do we interpret the following formulas:

- \( \beta(S^n 0) \)
  \( n \in A \)
- \( \beta(S##\sigma) \)
  \( ##\sigma \in A \)
- \( \sigma \iff \beta(S##\sigma) \)
  \( \sigma \text{ is true iff } ##\sigma \in A \)

The fixed-point lemma gives us the surprising result that for any such formula \( \beta \), we can always find a sentence \( \sigma \) such that the last formula not only is true in \( N \), but is derivable from \( A_E \).

Church’s Thesis (Again)

Recall that Church’s thesis states a relation is decidable iff the relation is recursive.

We have just shown that every recursive relation is representable in the theory \( C_n A_E \). This means that \( Cn A_E \) must be powerful enough to represent any decision procedure.

Using techniques like those we have just seen (representability using Gödel numbers), it can be shown that any model of computation can be mirrored using \( Cn A_E \).

In what follows, the terms recursive and decidable are used interchangeably.
Fixed-Point Lemma

We have \( h(#\alpha) = #(\alpha(S^{\#\alpha}0)) \),
\( \gamma \) defines \( \{#\alpha \mid h(#\alpha) \text{ is in the set defined by } \beta \} \), and \( \sigma \equiv \gamma(S^{\#\gamma}0) \).

**Theorem**

Given any formula \( \beta \) in which only \( v_1 \) occurs free, we can find a sentence \( \sigma \) such that \( A_E \vdash [\sigma \leftrightarrow \beta(S^{\#\sigma}0)] \).

**Proof**

Let \( \theta(v_1, v_2) \) functionally represent \( CnA_E \); a function \( h \) whose value at \( \langle#\alpha\rangle \) is \( #(\alpha(S^{\#\alpha}0)) \).

Let \( Sb(#\alpha, #x, #t) = #(\alpha^t) \).

Let \( f(n) = #(S^n0) \).

Let \( g(#\alpha, n) = Sb(#\alpha, #v_1, f(n)) = #(\alpha(S^n0)) \).

Let \( h(#\alpha) = g(#\alpha, #\alpha) = #(\alpha(S^{\#\alpha}0)) \).

Consider the formula \( \gamma \equiv \forall v_2 [\theta(v_1, v_2) \rightarrow \beta(v_2)] \).

This defines in \( N \) the set of all \( #\alpha \) such that \( h(#\alpha) \) is in the set defined by \( \beta \).

Now, let \( \sigma \equiv \gamma(S^{\#\gamma}0) \).

Gödel Incompleteness Theorem

**Theorem**

If \( A \subseteq ThN \) and \( #A \) is recursive, then \( CnA \) is not a complete theory.

**Proof**

Suppose \( CnA \) is complete. Since \( A \subseteq ThN \), it follows that \( CnA \subseteq ThN \). And since \( CnA \) is complete and \( ThN \) is satisfiable, we must have \( CnA = ThN \). But then \( #CnA = #ThN \) is recursive (by item 21). But every recursive set is definable in \( N \), and we just showed that \( #ThN \) is not definable in \( N \).

Hence, \( ThN \) cannot be axiomatized.

**Alternate Version**

Assume that \( #\Sigma \) is recursive and \( \Sigma \cup A_E \) is consistent. Then \( Cn\Sigma \) is not complete.

**Proof**

This is a corollary of the next theorem, because a complete axiomatizable theory must be recursive.

Tarski Undefinability Theorem

**Theorem**

The set \( #ThN \) is not definable in \( N \).

**Proof**

Suppose that \( \beta \) were a formula which defined the set \( #ThN \). Applying the fixed-point lemma to \( \neg \beta \), we get a sentence \( \sigma \) such that

\[ \models_N [\sigma \leftrightarrow \neg\beta(S^{\#\sigma}0)] \]

and thus,

\[ \models_N [\sigma \iff \neg\models_N \beta(S^{\#\sigma}0)] \]

So, if \( \sigma \in ThN (\models_N \sigma) \), then its Gödel number is not in the set \( \beta \) defines, meaning that \( \beta \) cannot define \( #ThN \).

On the other hand, if \( \sigma \not\in ThN (\neg\models_N \sigma) \), then its Gödel number is in the set \( \beta \) defines, meaning that \( \beta \) cannot define \( #ThN \).
General Undecidability Results

The previous theorem has significant consequences in terms of what is decidable in first-order logic.

Church's Theorem

The set of Gödel numbers of valid sentences (in the language of $N$) is not recursive.

Proof

In the strong undecidability theorem, take $T$ to be the set of valid sentences. □

Thus, the language of $N$ is an example of a first-order language whose valid sentences cannot be decided by any effective procedure.

In fact, it is known that a single two-place predicate symbol is sufficient for the set of valid formulas of a language to be undecidable.

Strong Undecidability of $Cn A_E$

Theorem

Let $T$ be any theory (in the language of $N$) such that $T \cup A_E$ is consistent. Then $\#T$ is not recursive.

Proof

Let $T''$ be the theory $Cn(T \cup A_E)$. If $\#T$ is recursive, then it is not hard to show given our results on representability that $\#T''$ is also recursive.

This means that there is a formula $\beta$ which represents $\#T'$ in $Cn A_E$. By the fixed-point lemma, there is a sentence $\sigma$ such that

$$A_E \vdash [\sigma \leftrightarrow \neg \beta(\#^\sigma 0)].$$

Essentially, $\sigma$ says “I am not in $T''$.” As before, this leads to a contradiction. □