Review

Last time

- Soundness
- Homomorphisms
- Completeness (first half)
Outline

- Completeness (second half)
- Compactness
- Definability of Models
- Enumerability Theorem
- Finite Models

Sources:

Sections 2.5 through 2.6 of Enderton.

Completeness

Completeness Theorem (Gödel, 1930)

If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

This is equivalent to the following statement: any consistent set of formulas is satisfiable.
Compactness

Compactness Theorem

If every finite subset $\Gamma_0$ of $\Gamma$ is satisfiable, then $\Gamma$ is satisfiable.

Proof

Suppose every finite subset $\Gamma_0$ of $\Gamma$ is satisfiable. By soundness, every finite subset is consistent. But since deductions are finite, it follows that $\Gamma$ is consistent. Thus, by completeness, $\Gamma$ is satisfiable.

Corollary

If $\Gamma \models \phi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \models \phi$.

Proof

Suppose to the contrary that $\Gamma_0 \nvdash \phi$ for every finite $\Gamma_0 \subseteq \Gamma$. Then every finite subset of $\Gamma \cup \{\neg \phi\}$ is satisfiable, and thus $\Gamma \cup \{\neg \phi\}$ is satisfiable. It follows that $\Gamma \nvdash \phi$. 
Reasonable Languages

A *reasonable* language is one whose signature can be effectively enumerated and such that the two relations

\[
\{ \langle P, n \rangle \mid P \text{ is an } n\text{-ary predicate symbol} \}
\]

and

\[
\{ \langle f, n \rangle \mid f \text{ is an } n\text{-ary function symbol} \}
\]

are decidable.

Any language constructed from a finite signature is reasonable. A reasonable language must be countable.
Enumerability Theorem

Theorem

For a reasonable language, if $\Gamma$ is a decidable set of formulas, then the set of all theorems of $\Gamma$ is effectively enumerable.

Proof

First note that for a reasonable language, the set $\Lambda$ of axioms is decidable. Given an expression, we can effectively check whether it is well-formed and whether it is a tautology or a syntactic instance of any of the other axiom groups.

Recall from lecture 2 that the set of tautological consequences of an effectively enumerable set is effectively enumerable. But as we proved earlier, $\phi$ is a tautological consequence of $\Gamma \cup \Lambda$ iff $\Gamma \vdash \phi$. And $\Gamma \vdash \phi$ iff $\Gamma \models \phi$ by soundness and completeness.
Corollaries

Corollary

The set of valid formulas in a reasonable language is effectively enumerable.

Corollary

If $\Gamma$ is a decidable set of formulas in a reasonable language and for any sentence $\sigma$, either $\Gamma \models \sigma$ or $\Gamma \models \neg \sigma$, then the set of sentences implied by $\Gamma$ is decidable.

Proof

Enumerate the theorems of $\Gamma$ until either $\sigma$ or $\neg \sigma$ is obtained.

\qed
Definability of a Class of Models

A *theory* is a set of sentences. For a given signature $\Sigma$, a $\Sigma$-theory is a set of sentences, each of which is a $\Sigma$-formula.

If $\mathcal{K}$ is a class of models with signature $\Sigma$, we say that a $\Sigma$-theory $T$ *axiomatizes* $\mathcal{K}$ or is a *set of axioms* for $\mathcal{K}$ if $\mathcal{K}$ is the class of all models of $T$.

For an arbitrary $\Sigma$-theory $T$, let $\text{Mod} T$ be the class of all models of $T$ (over signature $\Sigma$).

A class $\mathcal{K}$ of models is *first-order definable*, also known as an *elementary class* (EC), iff $\mathcal{K} = \text{Mod} \tau$ for some sentence $\tau$.

A class $\mathcal{K}$ of models is *first-order axiomatizable*, or *generalized first-order definable*, also known as an *elementary class in the wider sense* (EC$_\Delta$), iff $\mathcal{K} = \text{Mod} T$ for some set of sentences $T$. 
Example

Consider a signature \((P)\) with a single binary predicate symbol \(P\).

A model \((A, R)\) is an ordered set iff \(R\) is transitive and satisfies the trichotomy condition (which states that for any \(a, b \in A\) exactly one of \(\langle a, b \rangle \in R, a = b, \langle b, a \rangle \in R\) holds).

This can be written as a first-order sentence as follows:

\[
\forall x \forall y \forall z \left( Pxy \rightarrow Pyz \rightarrow Pxz \right) \land \\
\forall x \forall y \left( Pxy \lor x = y \rightarrow Pyx \right) \land \\
\forall x \forall y \left( Pxy \rightarrow \neg Pyx \right).
\]

Because these can be be translated into a sentence, the class of (nonempty) ordered sets is first-order definable.
Example

Consider a signature \((\circ)\) with a single binary function symbol \(\circ\).

The class of all groups is defined by the following sentence:

\[
\forall x \forall y \forall z (x \circ (y \circ z) = (x \circ y) \circ z) \land \\
\forall x \forall y \exists z (x \circ z = y) \land \\
\forall x \forall y \exists z (z \circ x = y).
\]

The class of all infinite groups is first-order axiomatizable. To see this, let

\[
\lambda_2 = \exists x \exists y x \neq y, \\
\lambda_3 = \exists x \exists y \exists z (x \neq y \land x \neq z \land y \neq z), \\
\ldots \\
\lambda_n = \text{there are at least } n \text{ distinct objects}
\]

Then the class of infinite groups is axiomatized by the sentence for groups together with the set of sentences \(\{\lambda_2, \lambda_3, \ldots\}\).

The class of infinite groups is not first-order definable.
Finite Models

Theorem

If a set $\Gamma$ of sentences has arbitrarily large finite models, then it has an infinite model.

Proof

For each integer $k \geq 2$, we can find a sentence $\phi_k$ which translates, “there are at least $k$ things”. For example,

$$\phi_2 = \exists v_1 \exists v_2 (v_1 \neq v_2)$$
$$\phi_3 = \exists v_1 \exists v_2 \exists v_3 (v_1 \neq v_2 \land v_1 \neq v_3 \land v_2 \neq v_3)$$

Now, consider the set $\Gamma \cup \{\phi_2, \phi_3, \ldots\}$. By hypothesis, any finite subset has a model. So by compactness the entire set has a model, which clearly must be infinite.
Finite Models

Corollary

The class $\mathcal{K}_f$ of all finite models (for a fixed signature) is not $\text{EC}_\Delta$, i.e. there is no set of sentences $\Gamma$ such that $\mathcal{K}_f = \text{Mod} \Gamma$.

Proof

Suppose such a set $\Gamma$ existed. Then since $\Gamma$ has models of arbitrarily large finite size, it must also have an infinite model, which is a contradiction.

\[
\square
\]

Corollary

The class of all infinite models is $\text{EC}_\Delta$ but not $\text{EC}$.

Proof

The set $\phi_2, \phi_3, \phi_4, \ldots$ is a first-order axiomatization of the class of all infinite models. Suppose that the class is EC. Then it is equal to $\text{Mod} \tau$ for some first-order sentence $\tau$. But then $\mathcal{K}_f = \text{Mod} \neg \tau$ and we know that $\mathcal{K}_f$ is not even $\text{EC}_\Delta$.

\[
\square
\]