Consider the following situation: There is a network of links between nodes — that is, an undirected graph whose vertices are the nodes and whose edges are the links. Each link fails with probability $P$. You wish to know what is the probability that $K$ pairs of nodes are connected, for $K$ between 0 and $N(N-1)/2$ where $N$ is the number of nodes.

**Part A**

Write a program `NumConnectedPairs(E,P)` that takes as input a symmetric array $E$ of 1’s and 0’s corresponding to the network and a probability $P$ that any given link will fail. It should return a probability distribution $D$ such that $D[i]$ is the probability that exactly $i$ pairs of cities are connected. This omits the probability that no cities are connected, which is always just $P^L$ where $L$ is the number of links.

There is no polynomial-time solution to this problem, so you should just use exhaustive enumeration; consider all possible subsets of the links, and for each, compute their probability and the number of pairs of nodes connected, and add up the total probability for each.

For a given subnetwork, the number of pairs connected can be computed as follows: Use a depth-first search to divide the network into connected components. Then each connected component of size $K$ connects $K(K-1)/2$ pairs of nodes.

For instance, consider the simple network of three cities shown below and suppose that each link can fail with probability 0.2.

![Network Diagram]

The corresponding function call would be

```python
E=[0,1,0,0; 1,0,1,0; 0,1,0,1; 0,0,1,0];
P=0.2;
NumConnectedPairs(E,P)
```

Since there are three links, there are 8 possible sub-networks. These are shown in the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Active links</th>
<th>Pairs connected</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1-2, 2-3, 3-4</td>
<td>6</td>
<td>0.512</td>
</tr>
<tr>
<td>2.</td>
<td>1-2, 2-3</td>
<td>3</td>
<td>0.128</td>
</tr>
<tr>
<td>3.</td>
<td>1-2, 3-4</td>
<td>2</td>
<td>0.128</td>
</tr>
<tr>
<td>4.</td>
<td>1-2</td>
<td>1</td>
<td>0.032</td>
</tr>
<tr>
<td>5.</td>
<td>2-3, 3-4</td>
<td>3</td>
<td>0.128</td>
</tr>
<tr>
<td>6.</td>
<td>2-3</td>
<td>1</td>
<td>0.032</td>
</tr>
<tr>
<td>7.</td>
<td>3-4</td>
<td>1</td>
<td>0.032</td>
</tr>
<tr>
<td>8.</td>
<td>∅</td>
<td>0</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Therefore the probability is 0.096 that 1 pair of cities is connected; 0.128 that 2 pairs are connected; 0.256 that 3 pairs are connected; and 0.512 that 6 pairs are connected. So the function should return [0.096, 0.128, 0.256, 0, 0, 0.512]

Part B

In the same situation as part A, write a function \( \text{ProbConnected}(E, P, \text{PairA}) \) where \( E \) and \( P \) are the same as in part A, and \( \text{PairA} \) is a pair of nodes. For instance, in the above example with the same values of \( E \) and \( P \), \( \text{ProbConnected}(E, P, [1, 2]) \) should return 0.8 and \( \text{ProbConnected}(E, P, [1, 4]) \) should return 0.512.

In this case the probability is easily computed, but that will not be the case in general; you should carry out the same enumeration of cases as in part A.

Part C

In the same situation as part A, write a function \( \text{CondProbConnected}(E, P, \text{PairA}, \text{PairB}, \text{BConn}) \) where \( E \) and \( P \) are the same as in part A; and \( \text{PairA} \) and \( \text{PairB} \) are each a pair of nodes. If \( \text{BConn}=1 \) then the function returns the conditional probability that \( \text{PairA} \) is connected given that \( \text{PairB} \) is connected; if \( \text{BConn}=0 \) then the function returns the conditional probability that \( \text{PairA} \) is connected given that \( \text{PairB} \) is not connected.

For instance, in the above example
- \( \text{CondProbConnected}(E, P, [1, 2], [1, 4], 1) \) should return 1.
- \( \text{CondProbConnected}(E, P, [1, 2], [1, 4], 0) \) should return 0.5902 (Cases 2, 3, 4 out of cases 2-8).
- \( \text{CondProbConnected}(E, P, [1, 4], [1, 2], 1) \) should return 0.64.
- \( \text{CondProbConnected}(E, P, [1, 4], [1, 2], 0) \) should return 0.
- \( \text{CondProbConnected}(E, P, [2, 4], [1, 3], 1) \) should return 0.8.
- \( \text{CondProbConnected}(E, P, [2, 4], [1, 3], 0) \) should return 0.3556 (Case 5 out of cases 3-8).

For all three parts of this problem it is OK to write an exponential space solution (i.e. one that essentially generates the table above as a data structure), but it is certainly better to write code that requires only reasonable amounts of memory, though an exponential amount of time.