1 How to Compute the Projection of a Point on a Line

Suppose the line ℓ is defined by two points p_0 and p_1. Let q be another point, which we must assume is not on ℓ. Our goal is to compute the point p on ℓ that is closest to q. This point p is called the **projection** of q on ℓ.

So write $p = (X, Y)$ where X, Y are unknowns. We just have to set up two equations that p must satisfy.

(1) The first equation says that the vector $p - q$ and $p_1 - p_0$ are perpendicular to each other. In general, two vectors u, v are perpendicular if their **scalar product** $u \cdot v$ is zero. The scalar product is defined by

$$u \cdot v = (u.x \times v.x) + (u.y \times v.y).$$

Hence our first equation is

$$(p - q) \cdot (p_1 - p_0) = 0. \tag{1}$$

(2) The second equation says that p lies on the line ℓ. This amounts to our orientation2d predicate discussed in class: $orientation2d(p_0, p_1, p) = 0$. In terms of 2×2 determinants, this is the same as

$$\det [p_1 - p_0, p - p_0] = 0. \tag{2}$$

Finally, let us put the above two equations into a more verbose form: First, Equation (1) is expanded into

$$\begin{aligned} (X - q.x)(p_1.x - p_0.x) + (Y - q.y)(p_1.y - p_0.y) &= 0 \\ X(p_1.x - p_0.x) + Y(p_1.y - p_0.y) - q.x(p_1.x - p_0.x) - q.y(p_1.y - p_0.y) &= 0 \end{aligned}$$

Next, Equation (2) becomes

$$\begin{aligned} (p_1.x - p_0.x)(Y - p_0.y) - (p_1.y - p_0.y)(X - p_0.x) &= 0 \\ -X(p_1.y - p_0.y) + Y(p_1.x - p_0.x) - p_0.y(p_1.x - p_0.x) + p_0.x(p_1.y - p_0.y) &= 0 \end{aligned}$$

Rewriting both equations as a single matrix equation,

$$\begin{bmatrix} p_1.x - p_0.x & p_1.y - p_0.y \\ p_0.y - p_1.y & p_1.x - p_0.x \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = - \begin{bmatrix} -q.x(p_1.x - p_0.x) - q.y(p_1.y - p_0.y) \\ -p_0.y(p_1.x - p_0.x) + p_0.x(p_1.y - p_0.y) \end{bmatrix}$$

This has the usual $\mathbf{Ax} = \mathbf{b}$ form of linear systems of equations.