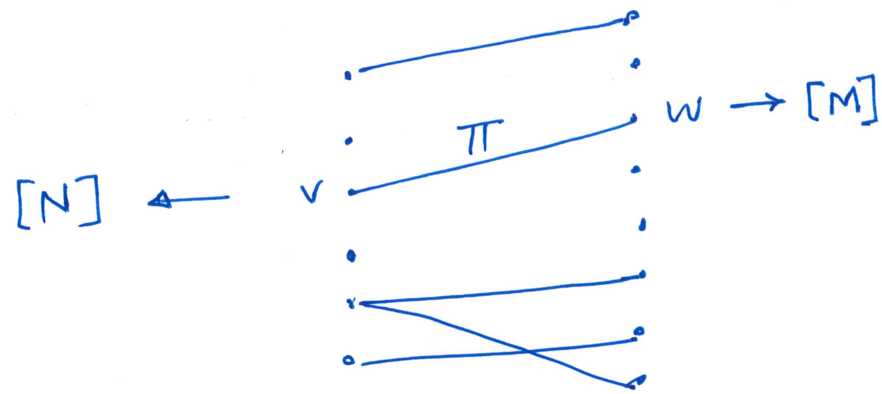
Label Cover

$$\mathcal{L}(G(V,W,E),\ [N],\ [M],\ \{\pi_{vw}\}).$$



$\pi = \pi_{vw}.$

$L \in NP.$ NP-hard to tell if

$x \in L \Rightarrow$ $OPT(\mathcal{L}) = 1$

$x \notin L \Rightarrow$ $OPT(\mathcal{L}) \leq \delta.$

"Outer" PCP. Proof is supposed to be labeling ℓ to $\mathcal{L}$.

- $\ell : V \to [N]$, $\ell : W \to [M]$.

Verifier

- Pick edge (v,w) at random.
- Read $\ell(v)$, $\ell(w)$.
- Accept iff $\pi(\ell(w)) = \ell(v)$.

1. 2-queries, from alphabet $[N]$, $[M]$.
2. Perfect completeness.
3. Soundness $\leq \delta$.

Achieving Boolean queries

Idea: - Proof composition

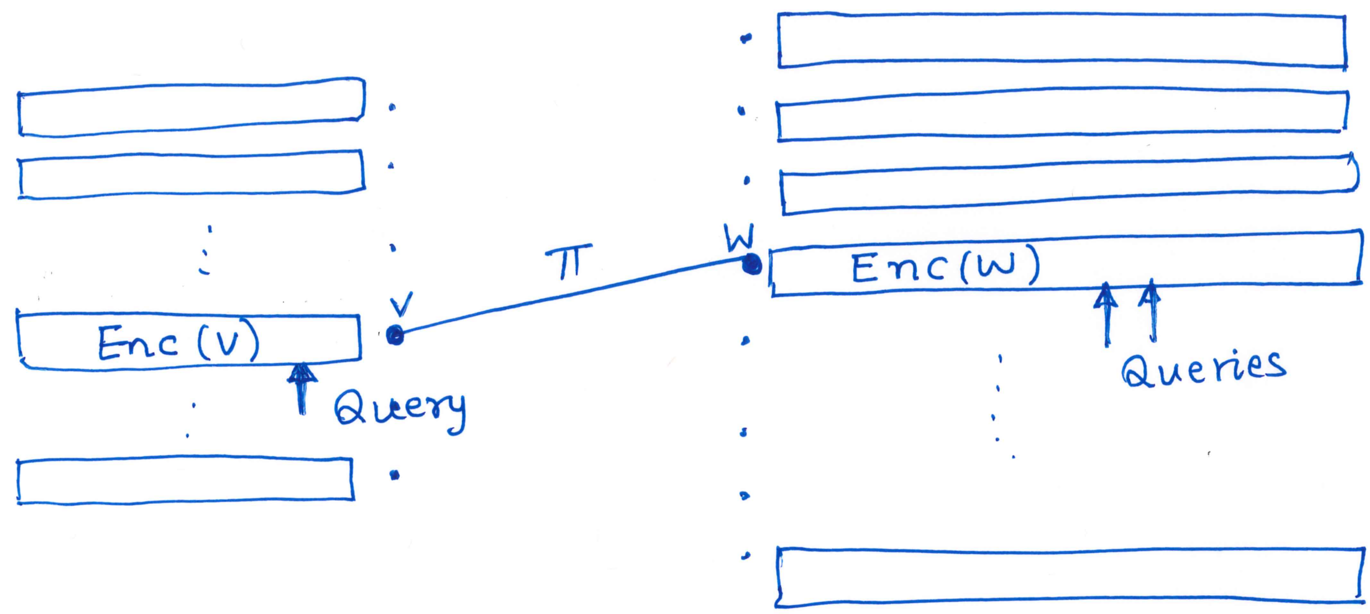
- Instead of labels (answers) $\ell(v), \ell(w)$ ask for

$$\text{LongCode}(\ell(v)) \quad 1 \leq \ell(v) \leq N$$
$$\text{LongCode}(\ell(w)) \quad 1 \leq \ell(w) \leq M.$$

- Let $Enc(v)$, $Enc(w)$ be these "supposed" long codes.

- Run $O(1)$-bit test on $Enc(v)$, $Enc(w)$.

Proof looks like



<u>Verifier needs to check that</u>

① $Enc(v)$, $Enc(w)$ are corret encodings, namely, $LongCode(v)$, $LongCode(w)$ resp. → "codeword test"

② $Enc(v)$ encodes $1 \le i_0 \le N$
$Enc(w)$ encodes $1 \le j_0 \le M$ <u>and</u>

$\pi(j_0) = i_0$. → "consistency test"

[Håstad] both in one shot !!

Consistency Testing (A, B, π)

Given two tables $A : \{-1,1\}^n \to \{-1,1\}$

$B : \{-1,1\}^m \to \{-1,1\}$

and $\pi : [m] \to [n]$

Verify that

- $A = \chi_{\{i_0\}}$ for some $1 \leq i_0 \leq n$
- $B = \chi_{\{j_0\}}$ for some $1 \leq j_0 \leq m$
- $\pi(j_0) = i_0$.

Keep in mind

- If $m = n$, $\pi = id$, $A = B$

  then this is same as Dictatorship testing.

- Recall. $Pr[\text{Accept}] \geq \frac{1}{2} + \eta$ then

  $\exists\, \alpha_0$ s.t. $|\hat{A}_{\alpha_0}| \geq \Omega(\eta)$, $|\alpha_0| \leq O(1/\varepsilon \log(1/\eta))$.

__Theorem__ Fix $\varepsilon, \eta > 0$.

There is a 3-bit test on (A, B, π)

- Reads one bit from A, two from B.

- If $A = \chi_{\{i_0\}}$, $\pi(j_0) = i_0$, then $\Pr[\text{Accept}] \geq 1-\varepsilon$.
  $B = \chi_{\{j_0\}}$,

- If $\Pr[\text{Acc}] \geq \frac{1}{2} + \eta$ then

$$\sum_{\substack{\alpha \subseteq \pi(\beta) \\ |\beta| \leq O(1/\varepsilon \log(1/\eta))}} \hat{A}_\alpha^2 \, \hat{B}_\beta^2 \geq \eta^2.$$

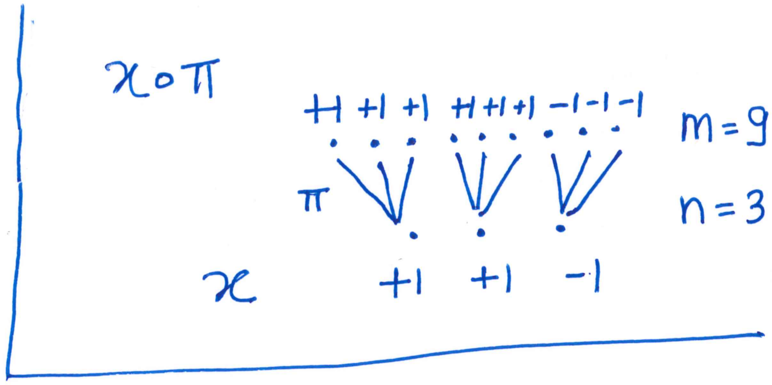
—— × ——

__Keep in mind__

- If $m = n$, $\pi = id$, $A = B$, this is

$$\sum_{\substack{\alpha \\ |\alpha| \leq O(1/\varepsilon \log(1/\eta))}} \hat{A}_\alpha^4 \geq \eta^2, \quad \text{i.e.}$$

A is $(\Omega(\eta), O(\frac{1}{\varepsilon}\log(\frac{1}{\eta})))$-junta.



Notation

For $x \in \{-1,1\}^n$,

$\pi : [m] \to [n]$,

let $x \circ \pi \in \{-1,1\}^m$ be such that

$$(x \circ \pi)_j = x_{\pi(j)} \quad \forall j \in \{1, 2, \dots, m\}.$$

Lemma $\beta \subseteq [m]$.

$$\chi_\beta(x \circ \pi) = \chi_{\pi_2(\beta)}(x).$$

where $\pi_2(\beta) := \{ i \in [n] \mid |\pi^{-1}(i) \cap \beta| \text{ is odd} \}$.

Proof.

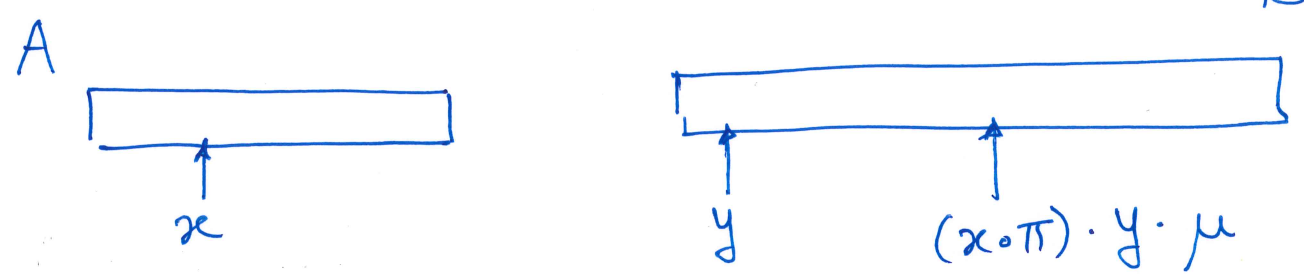
$$\chi_\beta(x \circ \pi) = \prod_{j \in \beta} (x \circ \pi)_j$$

$$= \prod_{j \in \beta} x_{\pi(j)}$$

$$= \prod_{i \in \pi_2(\beta)} x_i = \chi_{\pi_2(\beta)}(x).$$

$\blacksquare$

Test



- Pick $x \in \{-1,1\}^n$, $y \in \{-1,1\}^m$ uniformly at random.
- Pick $\mu \in \{-1,1\}^m$ as ε- noise.

$$\forall i \quad \mu_i = \begin{cases} +1 & \text{with prob. } 1-\varepsilon \\ -1 & \text{with prob } \varepsilon \end{cases}$$

- Accept iff

$$A(x)\ B(y) = B((x \circ \pi) \cdot y \cdot \mu).$$

Proof of completeness

Suppose $A = \chi_{\{i_0\}}$, $B = \chi_{\{j_0\}}$, $\pi(j_0) = i_0$.

Then

$$x_{i_0} \cdot \cancel{y_{j_0}} \stackrel{?}{=} (x \circ \pi)_{j_0} \cdot \cancel{y_{j_0}} \cdot \mu_{j_0}$$

$$\cancel{x_{i_0}} \stackrel{?}{=} \cancel{x_{\pi(j_0)}} \cdot \mu_{j_0}.$$

$$+1 \stackrel{?}{=} \mu_{j_0} \text{ (with prob } 1-\varepsilon).$$

Soundness $\quad \Pr[\text{Accept}]$

$$= \mathop{\mathbb{E}}_{x,y,\mu}\left[\frac{1 + A(x)\,B(y)\,B((x\circ\pi)\cdot y\cdot\mu)}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2}\mathop{\mathbb{E}}_{x,y,\mu}\left[\sum_{\alpha,\beta,\gamma}\hat{A}_\alpha\,\hat{B}_\beta\,\hat{B}_\gamma\;\chi_\alpha(x)\,\underline{\chi_\beta(y)}\;\chi_\gamma(x\circ\pi)\,\underline{\chi_\gamma(y)}\cdot\chi_\gamma(\mu)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\mathop{\mathbb{E}}_{x}\left[\sum_{\alpha,\beta}\hat{A}_\alpha\,\hat{B}_\beta^2\;\underbrace{\chi_\alpha(x)\,\chi_\beta(x\circ\pi)}_{\chi_\alpha(x)\,\chi_{\pi_2(\beta)}(x)}\cdot(1-2\varepsilon)^{|\beta|}\right] \qquad \beta=\gamma \qquad \alpha=\pi_2(\beta)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{\beta}\hat{A}_{\pi_2(\beta)}\,\hat{B}_\beta^2\,(1-2\varepsilon)^{|\beta|}.$$

$$\geq \frac{1}{2} + \eta$$

$$\therefore\; 2\eta \leq \sum_{\beta}\hat{A}_{\pi_2(\beta)}\,\hat{B}_\beta^2\,(1-2\varepsilon)^{|\beta|}.$$

By Cauchy-Schwartz $\quad \left(\sum_i p_i q_i \leq \sqrt{\sum_i p_i^2}\sqrt{\sum_i q_i^2}\right)$

$$2\eta \leqslant \sqrt{\sum_\beta \hat{A}^2_{\pi_2(\beta)} \hat{B}^2_\beta (1-2\varepsilon)^{2|\beta|}} \; \underbrace{\sqrt{\sum_\beta \hat{B}^2_\beta}}_{=1}$$

$$\therefore \; 4\eta^2 \leqslant \sum_\beta \hat{A}^2_{\pi_2(\beta)} \hat{B}^2_\beta (1-2\varepsilon)^{2|\beta|}.$$

- On R.H.S. contribution of those β for which $|\beta| \geqslant O(1/\varepsilon \log(1/\eta))$ is negligible (say $\leq \eta^2$).

- $$\boxed{\begin{array}{l} \therefore \; \sum_\beta \hat{A}^2_{\pi_2(\beta)} \hat{B}^2_\beta \geqslant \eta^2 \\ |\beta| \leq O(1/\varepsilon \log(1/\eta)) \end{array}}$$

as needed.

- We'll see a trick that ensures that $\beta \neq \phi$, $\pi_2(\beta) \neq \phi$.

$\blacksquare$