V22.0453-001: Honors Theory of Computation

Problem Set 1 Solutions

Problem 4 Give a regular expression for each of the following languages.

1. $\{w : \text{The length of } w \text{ is a multiple of } 3\}$.

   Solution: $(\Sigma\Sigma\Sigma)^*$

2. $\{w : w \text{ either starts with 01 or ends with 10}\}$.

   Solution: $(01\Sigma^*) \cup (\Sigma^*10)$

3. $\{w : w \text{ does not contain the substring 001}\}$.

   Solution: $(1 \cup 01)^*0^*$

Problem 6

The procedure for converting an NFA to an equivalent DFA given in class yields an exponential blowup in the number of states. That is, if the original NFA has n states, then the resulting DFA has 2^n states. In this problem, you will show that such an exponential blowup is necessary in the worst case.

Define $L_n = \{w : \text{The } n\text{th symbol from the right is } 1\}$.

1. Give an NFA with $n+1$ states that recognizes L_n.

2. Prove that any DFA with fewer than 2^n states cannot recognize L_n. (Hint: Let M be any DFA with fewer than 2^n states. Start by showing that there exist two different strings of length n that drive M to the same state.)

   Proof: Let M be any DFA with fewer than 2^n states. We will show that M cannot recognize L_n. Since there are 2^n strings of length n, by the Pigeonhole Principle, there are two different strings $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ that drive M to the same state. Since $x \neq y$, there is some i such that $x_i \neq y_i$. Without loss of generality, say that $x_i = 1$ and $y_i = 0$. Let $x' = x0^{i-1}$ and $y' = y0^{i-1}$. It is easy to see that $x' \in L_n$ and $y' \notin L_n$. However, since x and y drive M to the same state, it is clear that $x' = x0^{i-1}$ and $y' = y0^{i-1}$ also drive M to the same state, yet $x' \in L_n$ and $y' \notin L_n$. Therefore M cannot recognize L_n.