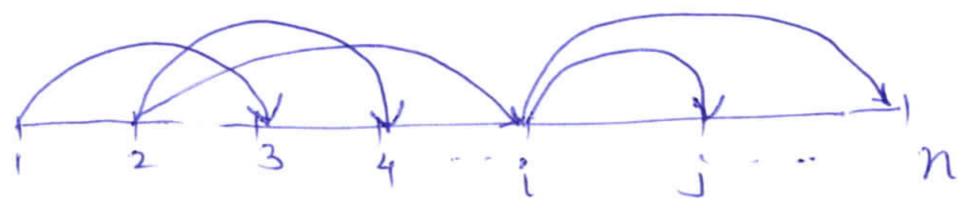
Structure Theorem for Directed Graphs

Def A directed graph $G(V, E)$ is called acyclic if it has no directed cycle.

Theorem #1 A directed graph $G(V, E)$ is acyclic iff its vertex set V can be labeled/ordered as $V = \{1, 2, \cdots, n\}$ so that

$$\forall (i,j) \in E, \quad i < j.$$



(i.e. its vertices can be ordered on a line so that all edges go forward).

Theorem #2 For an acyclic graph $G(V, E)$, the ordering as in Theorem #1 can be computed in $O(m+n)$ time given the adjacency list representation.

$m = |E|$
$n = |V|$

Such an ordering is called topological ordering.

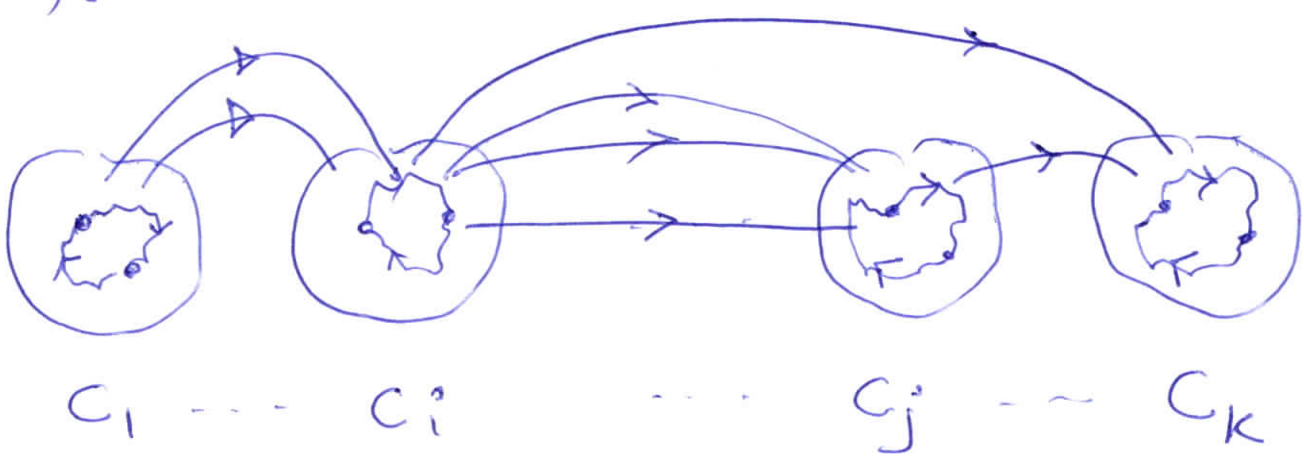
Def A directed graph $G(V,E)$ is called strongly connected if $\forall\ u,v \in V$, there is a directed $u \rightsquigarrow v$ path (and $v \rightsquigarrow u$ path).

Fact Given adjacency list representation, one can check if $G(V,E)$ is strongly connected in $O(m+n)$ time. (Using BFS)

Structure Theorem (Informally)

Every directed graph $G(V,E)$ can be
- decomposed into strongly connected components
- along with, acyclic (topological ordering) structure imposed on these components.

$G(V,E)$:



Theorem #3

For every directed graph $G(V,E)$, we can decompose V as $V = C_1 \cup C_2 \cup C_3 \cdots \cup C_k$ such that

- Every $G|_{C_i}$ is strongly connected. ($G|_{C_i}$ is restriction of vertices and edges to C_i).

- For all other edges $(a,b) \in E$, $a \in C_i$, $b \in C_j$ for some $i < j$.

Theorem #4 The decomposition into strongly connected components as in Theorem #3 can be computed in $O(m+n)$ time given adjacency list representation.

Strategy to design algorithms on directed graphs

① Design algorithm (in the special case) when the graph is acyclic. (often dynamic pgm.).

② On a general graph, first find its decomposition into strongly connected components as in Theorem # 3, 4.

③ And then combine / generalize Algorithm in step ① with / using the decomposition in step ②.